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HYDROMAGNETIC VISCOUS FLUID OVER A NON-LINEAR STRETCHING AND SHRINKING SHEET IN THE PRESENCE OF THERMAL RADIATION

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ABSTRACT

In this paper, the effects of suction/blowing and thermal radiation on a hydromagnetic viscous fluid over a non-linear stretching and shrinking sheet are investigated. A similarity transformation is used to reduce the governing equations to a set of nonlinear ordinary differential equations. The system of equations is solved analytically employing homotopy analysis method (HAM). Convergence of the HAM solution is checked. The resulting similarity equations are solved numerically using Matlab bvp4c numerical routine. It is found that dual solutions exist for this particular problem. The comparison of analytical solution and numerical solution for the velocity profile is an excellent agreement.
Keywords: Heat transfer; Homotopy analysis method; viscous fluid; Non-linear shrinking sheet; Thermal radiation

1. INTRODUCTION

Investigations of heat transfer and boundary layer flow on a continuously moving or stretching surface have important applications in many manufacturing processes and polymer industry, for examples, a continuous stretching of plastic films, artificial fibers, metal spinning, metal extrusion, continuous casting, glass blowing and many more. The pioneering work on the continuously stretching sheet was first initiated by Sakiadis (1961). The problem in Sakiadis (1961) is extended to discuss the various aspects of flow and heat transfer characteristics by many researchers like Dutta et al.(1985), Chen et al.(1988), Ali (1995), Liao (2005), Hayat et al.(2010), Makinde et al.(2013), Madhu et al.(2015) and Yasin et al.(2016).

The problem of the shrinking sheet where the velocity on the boundary is towards the origin or a fixed point, and the unsteady shrinking film solution was first investigated by Wang (1990). Again, Miklavcic and Wang (2006) studied the viscous hydrodynamic flow over a shrinking sheet for both two-dimensional and axisymmetric flows. It is also noted that the mass suction at the wall is required generally to maintain (or smooth) the flow over a shrinking sheet. They discussed the proof of existence and (non) uniqueness of both exact numerical and closed form solutions. The analysis of Miklavcic et al. (2006) was also extended in various directions for different fluids by many researchers such as Hayat et al.(2007), Kandasamy et al.(2008), Sajid et al. (2009), Fang et al.(2009), Noor et al.(2010) and Patil et al.(2016). Fang (2008) investigated the boundary layer flow over a shrinking sheet with surface moving with power-law velocity. Javed et al. (2011) investigated the boundary layer flow and heat transfer analysis of electrically conducting viscous fluid over a nonlinearly shrinking sheet. Recently, Bhattacharyya (2013) studied the heat transfer in unsteady boundary layer stagnation-point flow over a shrinking/stretching sheet.

The homotopy analysis method (HAM) is one of the well-known methods to solve non-linear equations that does not need to any small parameter. This method has been introduced by Liao (1992), (1995), (1997), (2003), Liao et al. (2003) and (2004). The method has been used by many authors like Hayat et al. (2004), Hayat et al.(2004), (2007) and (2008). Also, Mehmood et al. (2006) and (2008). Then, Liao (2009), Fakhari et al. (2007) and Domairry et al. (2008), Domairry et al. (2009). Then, Tan et al. (2008), Ali et al. (2008) and Ziabakhsh et al. (2009) in a wide variety of scientific and engineering applications to solve different types of governing differential equations: linear and non-linear, homogeneous and non-homogeneous, and coupled and decoupled as well. This method offers highly accurate successive approximations of the solution. Also, Abdelmeguid et al. (2007) studied the effect of chemical reaction, variable viscosity and radition. In this paper, we will study the effects of suction/blowing, power index parameter, Magnetic field, Prandtl number and thermal radiation on a hydromagnetic viscous fluid over a non-linear stretching and shrinking sheet. The system of nonlinear coupled ordinary differential equations is solved using homotopy analysis method (HAM).

2. PROBLEM FORMULATION

Consider a two-dimensional flow of an incompressible viscous fluid past a porous shrinking sheet at y = 0. It is assumed that the velocity of the stretching /shrinking sheet is $U_w(x) = \propto c\, x^m$, where $\propto = 1, -1$ is respectively for stretching and shrinking sheet. It is also assumed that constant mass transfer velocity is $v = v_w(x)$ with $v_w(x) < 0$ for suction and $v_w(x) > 0$ for injection, respectively. The x-axis is taken along the stretching/shrinking sheet and the y-axis perpendicular to it into the fluid. The fluid is electrically conducting and the magnetic field B(x) is assumed to be applied in the y-direction. The magnetic Reynolds number is taken to be small so that the induced magnetic field can be neglected. The temperature of the surface maintained at a constant temperature T_w and far away from the sheet temperature is T_∞, where $T_w > T_\infty$. Under boundary layer approximation, the continuity, momentum, and energy equations are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0\,, \tag{1}$$

$$u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} = \nu\,\frac{\partial^2 u}{\partial y^2} - \frac{\sigma\, B^2(x)}{\rho}\,u\,, \tag{2}$$

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$$\rho\, c_p \left(u\,\frac{\partial T}{\partial x} + v\,\frac{\partial T}{\partial y}\right) = k\,\frac{\partial^2 T}{\partial y^2} + \mu\left(\frac{\partial u}{\partial y}\right)^2 - \frac{\partial q_r}{\partial y} \tag{3}$$

where u and v are the velocity components in the x- and y-directions, respectively, ρ is the fluid density, μ is the dynamic viscosity, $\nu = \frac{\mu}{\rho}$ is the kinematic viscosity, σ is the electrical conductivity of the fluid, T is the temperature, c_p is the specific heat at constant pressure, and k is the thermal diffusivity. In Eq. (2), the external electric field and the polarization effects are neglected and Chaim (1995) $B(x) = B_o\, x^{\frac{m-1}{2}}$. q_r is the radiative heat flux. Using Rosseland's approximation for radiation Brewster (1972), we obtain $q_r = \frac{-4\,\sigma_1}{3\,\alpha_R}\frac{\partial T^4}{\partial y}$, where σ_1 is the Stefan–Boltzmann constant, α_R is the absorption coefficient. We presume that the temperature variation within the flow is such that T^4 may be expanded in a Taylor's series. Expanding T^4 about T_∞ and neglecting higher order terms we get, $T^4 = 4\,T_\infty^3\,T - 3\,T_\infty^4$.
The appropriate boundary conditions for the velocity components and temperature are given by

$$u = \alpha\, U_w(x) = \alpha\, c\, x^m, \quad v = v_w(x),$$

$$T = T_w \quad \text{at} \quad y = 0 \tag{4}$$

$$u \to 0\,, \quad T \to T_\infty \quad \text{as} \quad y \to \infty \tag{5}$$

where m is a power index and c is a constant rate stretching/shrinking which has a dimension of $(\text{time})^{-1}$.
We are interested in obtaining a similarity solution of the form

$$\eta = y\sqrt{\frac{c\,(m+1)}{2\,\nu}}\; x^{\frac{m-1}{2}}\,,$$

$$u = c\, x^m\, f'(\eta)$$

$$v = -\sqrt{\frac{c\,\nu\,(m+1)}{2}}\; x^{\frac{m-1}{2}}\left[f(\eta) + \frac{m-1}{m+1}\,\eta\, f'(\eta)\right],$$

$$\theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty} \tag{6}$$

and the wall mass transfer velocity becomes Fang (2008)

$$v_w(x) = -\sqrt{\frac{c\,\nu\,(m+1)}{2}}\; x^{\frac{m-1}{2}}\, f(0) \;\propto\; x^{\frac{m-1}{2}}. \tag{7}$$

Using Eq. (6), the continuity Eq. (1) is identically satisfied and Eqs.(2) and (3) reduce to the following ordinary differential equations

$$f''' + f\,f'' - \beta\,(f')^2 - M\,f' = 0\,, \tag{8}$$

$$\left(1 + \frac{1}{R}\right)\theta'' + Pr\left[f\,\theta' + Ec\,(f'')^2\right] = 0\,. \tag{9}$$

subject to the boundary conditions

$$f'(0) = \alpha\,, \quad f(0) = s\,, \quad \theta(0) = 1\,, \tag{10}$$

$$f'(\infty) = 0\,, \quad \theta(\infty) = 0\,. \tag{11}$$

where primes denote differentiation with respect to η. The control parameter β, the magnetic field or Hartman number M, the thermal radiation parameter R, the Prandtl number Pr, the Eckert number Ec and the wall mass transfer at the sheet s are given by

$$\beta = \frac{2\,m}{m+1}\,, \quad M = \frac{2\,\sigma\, B_o^2}{\rho\, c\,(m+1)}\,,$$

$$R = \frac{3\,\alpha_R\,k}{16\,\sigma_1\,T_\infty^3}\,, \quad Pr = \frac{\mu\, c_p}{k}\,,$$

$$Ec = \frac{U_w^2}{c_p\,(T_w - T_\infty)} \quad \text{and}$$

$$s = -\sqrt{\frac{2}{c\,\nu\,(m+1)}}\,. \tag{12}$$

The physical quantities of interest in this problem are the skin-friction parameter c_f and local Nusselt number Nu which are defined by

$$c_f = \frac{\tau_w}{\rho\, U_w^2} = \sqrt{\frac{m+1}{2}}\,(Re)^{\frac{-1}{2}}\, f''(0)\,, \tag{13}$$

where the wall shear stress τ_w and the local Reynolds number Re are given by

$$\tau_w = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} = c\,\mu\sqrt{\frac{c\,(m+1)}{2\,\nu}}\; x^{\frac{3m-1}{2}}\, f''(0)$$

$$\text{and} \qquad Re = \frac{U_w\, x}{\nu}\,. \tag{14}$$

The local rate of heat transfer of the surface is

$$q_w = -k\left(\frac{\partial T}{\partial y}\right)_{y=0} = -k\,(T_w - T_\infty)\sqrt{\frac{c\,(m+1)}{2\,\nu}}\; x^{\frac{m-1}{2}}\,\theta'(0)\,, \tag{15}$$

which can be used to compute the local Nusselt number

$$Nu = \frac{x\, q_w}{k\,(T_w - T_\infty)} = -\,(Re)^{\frac{1}{2}}\sqrt{\frac{m+1}{2}}\;\theta'(0)\,. \tag{16}$$

3. HOMOTOPY ANALYSIS SOLUTION

3.1. ZERO-ORDER DEFORMATION EQUATIONS

Solving Eqs. (8)–(11) using HAM (Hayat (2007), Hang (2007), Zhu (2009), Ali (2008) and Ziabakhsh (2009)). From the boundary conditions (10) and (11), it is obvious to choose:

$$f_o(\eta) = s + 1 - e^{-\alpha\,\eta}\,, \tag{17}$$

$$\theta_o(\eta) = e^{-\eta}\,, \tag{18}$$

as the initial approximations of $f(\eta)$ and $\theta(\eta)$, respectively, and to choose:

$$L_f[\, f(\eta\,;q)\,] = \frac{\partial^3\Phi(\eta\,;q)}{\partial\eta^3} + \frac{\partial^2\Phi(\eta\,;q)}{\partial\eta^2}\,, \tag{19}$$

$$L_\theta[\,\phi(\eta\,;q)\,] = \frac{\partial^2\Theta(\eta\,;q)}{\partial\eta^2} + \frac{\partial\Theta(\eta\,;q)}{\partial\eta}\,, \tag{20}$$

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as the auxiliary linear operators, which have the following properties:

$$L_f[\, c_1 + c_2\, \eta + c_3\, e^{-\eta}\,] = 0\,,$$

$$L_\theta[\, c_4 + c_5\, e^{-\eta}\,] = 0. \tag{21}$$

where $c_i\ (i = 1 - 5)$ are arbitrary constants. Based on (8) and (9), This paper is led to define the non-linear operators:

$$N_f[\Phi(\eta;q)] = \frac{\partial^3\Phi(\eta;q)}{\partial\eta^3} + \Phi(\eta;q)\,\frac{\partial^2\Phi(\eta;q)}{\partial\eta^2} - \beta\left[\frac{\partial\Phi(\eta;q)}{\partial\eta}\right]^2 - M^2\,\frac{\partial\Phi(\eta;q)}{\partial\eta}\,, \tag{22}$$

$$N_\theta[\Theta(\eta;q)] = \left(1+\frac{1}{R}\right)\frac{\partial^2\Theta(\eta;q)}{\partial\eta^2} + \Pr\left[\Phi(\eta;q)\,\frac{\partial\Theta(\eta;q)}{\partial\eta} + Ec\left[\frac{\partial^2\Phi(\eta;q)}{\partial\eta^2}\right]^2\right]. \tag{23}$$

Let h denote the non-zero auxiliary parameter. Then construct the zeroth-order deformation equations:

$$(1-q)\, L_f[\Phi(\eta;q) - f_o(\eta)] = q\, h\, H_f(\eta)\, N_f[\Phi(\eta;q)] \tag{24}$$

$$(1-q)\, L_\theta[\Theta(\eta;q) - \theta_o(\eta)] = q\, h\, H_\theta(\eta)\, N_\theta[\Theta(\eta;q)] \tag{25}$$

Subject to the boundary conditions:

$$\Phi(0;q) = s\,,\quad \left.\frac{\partial\Phi(\eta;q)}{\partial\eta}\right|_{\eta=0} = \alpha\,,$$

$$\left.\frac{\partial\Phi(\eta;q)}{\partial\eta}\right|_{\eta=\infty} = 0\,, \tag{26}$$

$$\Theta(0;q) = 1\,,\quad \Theta(\infty;q) = 0\,. \tag{27}$$

where $q \in [\,0\,,1\,]$ is an embedding parameter. When $= 0$, it is straightforward that:

$$\Phi(\eta;0) = f_o(\eta)\,,\quad \Theta(\eta;0) = \theta_o(\eta)\,. \tag{28}$$

When $q = 1$ the zeroth-order deformation equations (24)–(27) are equivalent to the original equations (8)–(11), so that we have:

$$\Phi(\eta;q) = f(\eta)\,,\quad \Theta(\eta;q) = \theta(\eta)\,, \tag{29}$$

respectively. Thus as q increases from 0 to 1, $\Phi(\eta;q)$ and $\Theta(\eta;q)$ vary from the initial guess $f_o(\eta)$ and $\theta_o(\eta)$ to the solutions $f(\eta)$ and $\theta(\eta)$ of the problem, respectively. So expanding $\Phi(\eta;q)$ and $\Theta(\eta;q)$ in Taylor's series about the embedding parameter q, we have:

$$\Phi(\eta;q) = \Phi(\eta;0) + \textstyle\sum_{m=1}^{\infty} f_m(\eta)\, q^m\,, \tag{30}$$

$$\Theta(\eta;q) = \Theta(\eta;0) + \textstyle\sum_{m=1}^{\infty} \theta_m(\eta)\, q^m\,, \tag{31}$$

where:

$$f_m(\eta) = \frac{1}{m!}\left.\frac{\partial^m\Phi(\eta;q)}{\partial q^m}\right|_{q=0}\,, \tag{32}$$

$$\theta_m(\eta) = \frac{1}{m!}\left.\frac{\partial^m\Theta(\eta;q)}{\partial q^m}\right|_{q=0}\,. \tag{33}$$

If h is properly chosen, the series (30) and (31) are convergent at $q = 1$, we have, using (28) and (29), the solution series:

$$f(\eta) = f_o(\eta) + \sum_{m=1}^{\infty} f_m(\eta)\,, \tag{34}$$

$$\theta(\eta) = \theta_o(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta)\,. \tag{35}$$

3.2. HIGHER ORDER DEFORMATION EQUATIONS

Differentiating the zero-order deformation equations (24) and (25) m times about q, then setting $q = 0$, and finally dividing them by $m!$, we obtain the mth-order deformation equations:

$$L_f[\, f_m(\eta) - \chi_m\, f_{m-1}(\eta)\,] = h\, H_f(\eta)\, R_m(\eta)\,, \tag{36}$$

$$L_\theta[\, \theta_m(\eta) - \chi_m\, \theta_{m-1}(\eta)\,] = h\, H_\theta(\eta)\, S_m(\eta)\,, \tag{37}$$

subject to the boundary conditions:

$$f_m(0) = f'_m(0) = f'_m(\infty) = 0\,, \tag{38}$$

$$\theta_m(0) = \theta_m(\infty) = 0\,, \tag{39}$$

where

$$\chi_m = \begin{cases} 0\,; & m \le 1 \\ 1\,; & m \ge 2 \end{cases} \tag{40}$$

and

$$R_m(\eta) = f'''_{m-1}(\eta) + \sum_{k=0}^{m-1} f_k(\eta)\, f''_{m-1-k}(\eta) - \beta\sum_{k=0}^{m-1} f'_k(\eta)\, f'_{m-1-k}(\eta) - M^2\, f'_{m-1}(\eta), \tag{41}$$

$$S_m(\eta) = \left(1+\frac{1}{R}\right)\theta''_{m-1}(\eta) + \Pr\left[\sum_{k=0}^{m-1} f_k(\eta)\,\theta'_{m-1-k}(\eta) + Ec\sum_{k=0}^{m-1} f''_k(\eta)\, f''_{m-1-k}(\eta)\right]. \tag{42}$$

According to initial approximations and the auxiliary linear operators, we set:

$$H_f(\eta) = e^{-\eta}\,,\quad H_\theta(\eta) = e^{-\eta}\,. \tag{43}$$

The first order deformation equations:

$$L_f\,[f_1(\eta)] = h\, H_f(\eta)\, R_1(\eta)\,, \tag{44}$$

$$L_\theta[\, \theta_1(\eta)\,] = h\, H_\theta(\eta)\, S_1(\eta)\,, \tag{45}$$

and the boundary conditions:

$$f_1(0) = f_1'(0) = f_1'(\infty) = 0 , \quad (46)$$

$$\theta_1(0) = \theta_1(\infty) = 0 , \quad (47)$$

so that we have:

$$f_1(\eta) = a_1 e^{-(\alpha+1)\eta} + a_2 e^{-(2\alpha+1)\eta} + a_3 e^{-\eta} + a_4 , \quad (48)$$

$$\theta_1(\eta) = b_1 e^{-2\eta} + b_2 e^{-(\alpha+2)\eta} + b_3 e^{-(2\alpha+1)\eta} + b_4 e^{-\eta}, \quad (49)$$

where

$$a_1 = \frac{-h}{(\alpha+1)^2} \{ \alpha^2 - (s+1)\alpha - M^2 \} ,$$

$$a_2 = \frac{-h(1-\beta)\alpha}{2(2\alpha+1)^2} ,$$

$$a_3 = -\{ (\alpha+1) a_1 + (2\alpha+1) a_2 \} ,$$

$$a_4 = \alpha (a_1 + 2 a_2) ,$$

$$b_1 = \frac{h}{2} \left\{ \left(1+\frac{1}{R}\right) - Pr(s+1) \right\} ,$$

$$b_2 = \frac{h\,(Pr)}{(\alpha+1)(\alpha+2)} ,$$

$$b_3 = \frac{h\,(Pr)\,(Ec)\,\alpha^4}{(2\alpha)(2\alpha+1)} ,$$

$$b_4 = -(b_1 + b_2 + b_3) .$$

Similarly, we obtain:

$$f_2(\eta) = c_1 e^{-(\alpha+2)\eta} + c_2 e^{-(2\alpha+2)\eta} + c_3 e^{-2\eta} + c_4 e^{-(3\alpha+2)\eta} + c_5 e^{-(\alpha+1)\eta} + c_6 e^{-\eta} + c_7 , \quad (50)$$

$$\theta_2(\eta) = d_1 e^{-3\eta} + d_2 e^{-2\eta} + d_3 e^{-(\alpha+3)\eta} + d_4 e^{-(2\alpha+2)\eta} + d_5 e^{-(\alpha+2)\eta} + d_6 e^{-(2\alpha+3)\eta} + d_7 e^{-(3\alpha+2)\eta} + d_8 e^{-\eta} . \quad (51)$$

where

$$c_1 = \frac{-h}{(\alpha+1)(\alpha+2)^2} \{ (\alpha+1)[(\alpha+1)(s-\alpha) + M^2] a_1 - (\alpha^2+1) a_3 \} ,$$

$$c_2 = \frac{-h}{(2\alpha+1)(2\alpha+2)^2} \{(2\alpha+1)[(2\alpha+1)(s-2\alpha) + M^2] a_2 - [(\alpha+1)^2 + \alpha^2 - 2\beta\alpha(\alpha+1)] a_1\} ,$$

$$c_3 = \frac{-h}{4} (M^2 + s) a_3 ,$$

$$c_4 = \frac{-h}{(3\alpha+1)(3\alpha+2)^2} \{-(2\alpha+1)^2 - \alpha^2 + 2\beta\alpha(2\alpha+1)\} a_2 ,$$

$$c_5 = \frac{-h}{\alpha(\alpha+1)^2} (-\alpha^2 a_4) ,$$

$$c_6 = -\{ (\alpha+2) c_1 + (2\alpha+2) c_2 + 2c_3 + (3\alpha+2) c_4 + (\alpha+1) c_5 \} ,$$

$$c_7 = (\alpha+1) c_1 - \frac{1}{(2\alpha+2)^2} ,$$

$$d_1 = \frac{h}{6} \left\{ 4 b_1 \left(1+\frac{1}{R}\right) - 2b_1(s+1)\,Pr - a_3 \right\} ,$$

$$d_2 = \frac{h}{2} \left\{ b_4 \left(1+\frac{1}{R}\right) - b_4(s+1)\,Pr - a_4 \right\} ,$$

$$d_3 = \frac{h}{(\alpha+3)(\alpha+2)} \left\{ b_2 \left(1+\frac{1}{R}\right)(\alpha+2)^2 + 2b_1\,Pr - b_2(\alpha+2)(s+1)\,Pr - a_1 \right\} ,$$

$$d_4 = \frac{h}{(2\alpha+2)(2\alpha+1)} \left\{ b_3 \left(1+\frac{1}{R}\right)(2\alpha+1)^2 - b_3(2\alpha+1)(s+1)\,Pr - 2\,Ec\,a_1\,\alpha^2\,(\alpha+1)^2 \right\} ,$$

$$d_5 = \frac{h}{(\alpha+2)(\alpha+1)} \{ b_4\,Pr - 2\,Ec\,a_3\,\alpha^2 \} ,$$

$$d_6 = \frac{h}{(2\alpha+3)(2\alpha+2)} \{ b_2 (\alpha+2)\,Pr - a_2 \} ,$$

$$d_7 = \frac{h}{(3\alpha+2)(3\alpha+1)} \{ b_3 (2\alpha+1)\,Pr - 2\,Ec\,a_2\,\alpha^2 (2\alpha+1)^2 \} ,$$

$$d_8 = -\{ d_1 + d_2 + d_3 + d_4 + d_5 + d_6 + d_7 \} .$$

Since the solutions $f_3(\eta)$ and $\theta_3(\eta)$ are too long, so they are shown graphically.

4. CONVERGENCE OF THE HAM SOLUTION

For an analytic solution obtained by the homotopy analysis method, its convergent depends on the auxiliary parameter h. If this parameter is properly chosen, the given solution is valid, as verified in previous works Hayat et al. (2007), Hang et al. (2007), Zhu (2009), Ali et al (2008) and Ziabakhsh (2009). Since the interval for the admissible values of h corresponds to the line segments nearly parallel to the horizontal axis. Then, by plotting the curves of $f''(0)$ and $\theta'(0)$ versus h (which is called the h-curves of $f''(0)$ and $\theta'(0)$), we can obtain a valid region $-1.4 \leq h \leq -0.6$ from Figs. 1 and 2. Then h can be chosen in the region $[-1.4, -0.6]$. In this paper we choose $h = -1.2$.

5. RESULTS AND DISCUSSION

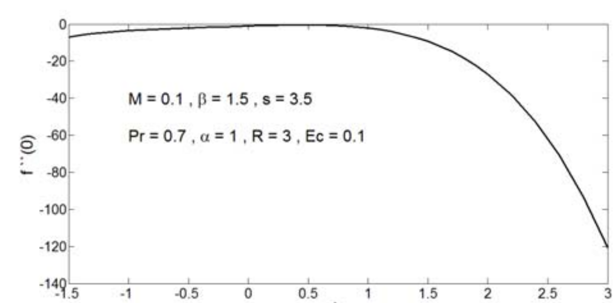

Fig. 1 The h-curves of $f''(0)$ obtained by the fifth-order approximation of the HAM

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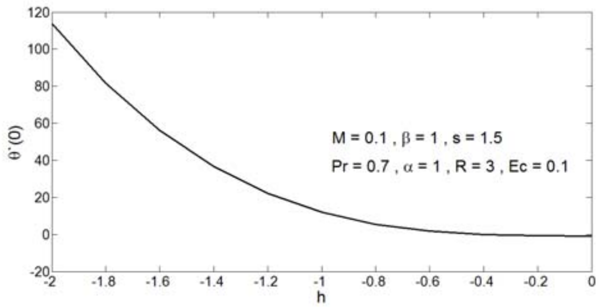

Fig. 2 The *h*-curves of $\theta^{`}(0)$ obtained by the fifth-order approximation of the HAM

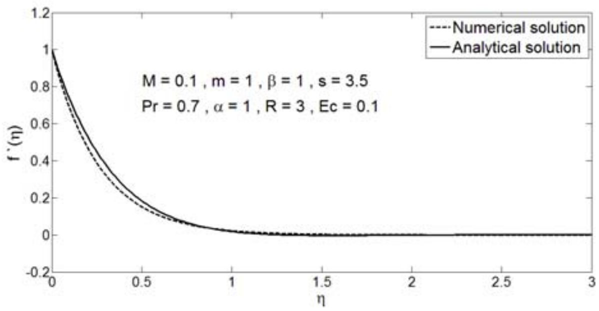

Fig. 3 Comparison of the velocity $f^{`}(\eta)$ between numerical and analytical solution

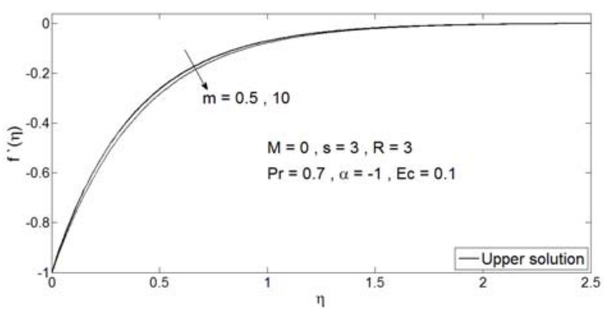

Fig. 4 The effects of power-index m of the surface velocity on the velocity $f^{`}(\eta)$. In case of shrinking sheet $\alpha = -1$. (Upper solution)

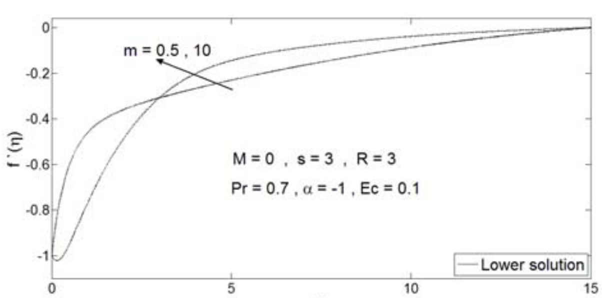

Fig. 5 The effects of power-index m of the surface velocity on the velocity $f^{`}(\eta)$. In case of shrinking sheet $\alpha = -1$. (Lower solution)

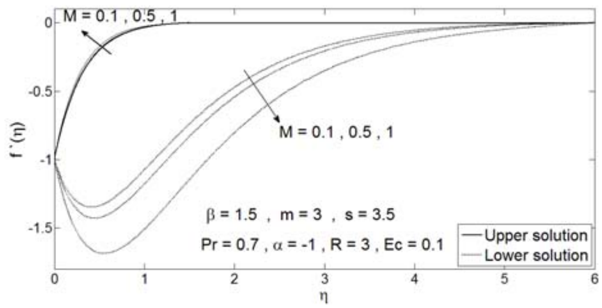

Fig. 6 The effects of the magnetic field M on the velocity $f^{`}(\eta)$. In case of shrinking sheet $\alpha = -1$

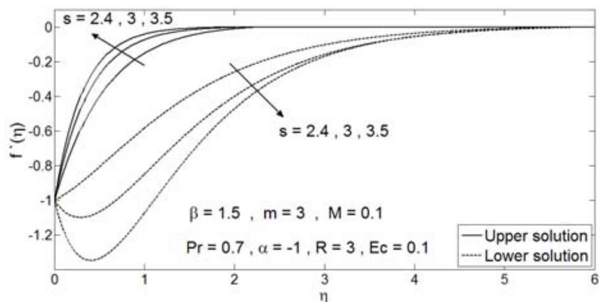

Fig. 7 The effects of mass suction s on the velocity $f^{`}(\eta)$. In case of shrinking sheet $\alpha = -1$

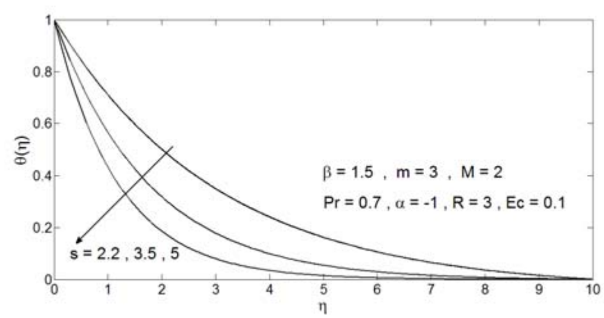

Fig. 8 The effects of mass suction s on the temperature field $\theta(\eta)$. In case of shrinking sheet $\alpha = -1$

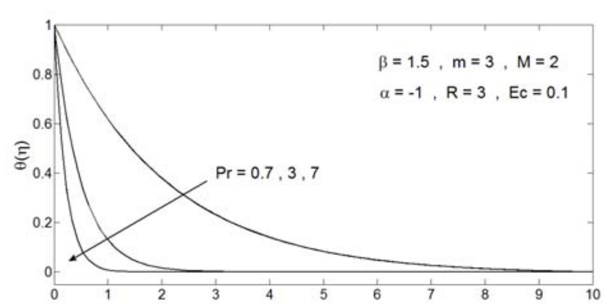

Fig. 9 The effects of the Prandtl parameter Pr on the temperature field $\theta(\eta)$. In case of shrinking sheet $\alpha = -1$

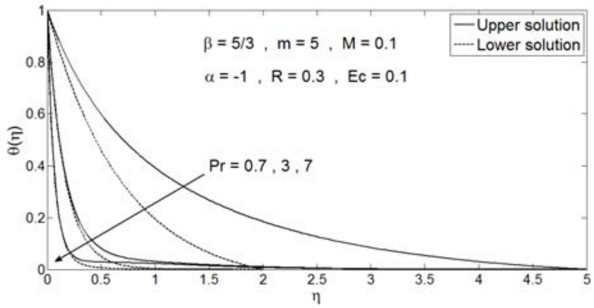

Fig. 10 The effects of the Prandtl parameter Pr on the temperature field $\theta(\eta)$. In case of shrinking sheet $\alpha = -1$ (Upper and Lower solution)

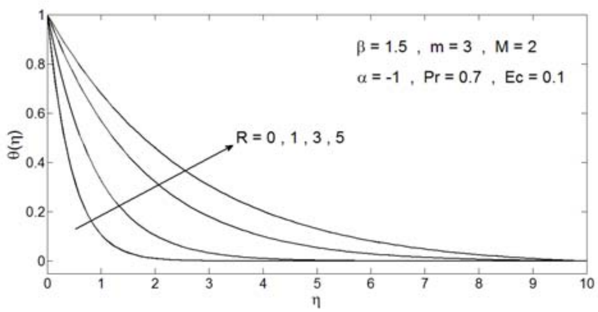

Fig. 11 The effects of the Radiation parameter R on the temperature field $\theta(\eta)$. In case of shrinking sheet $\alpha = -1$

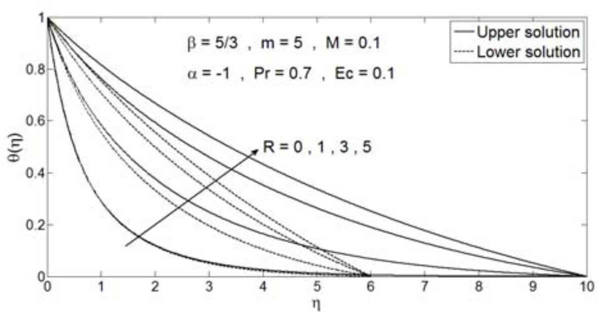

Fig. 12 The effects of the Radiation parameter R on the temperature field $\theta(\eta)$. In case of shrinking sheet $\alpha = -1$ (Upper and Lower solution)

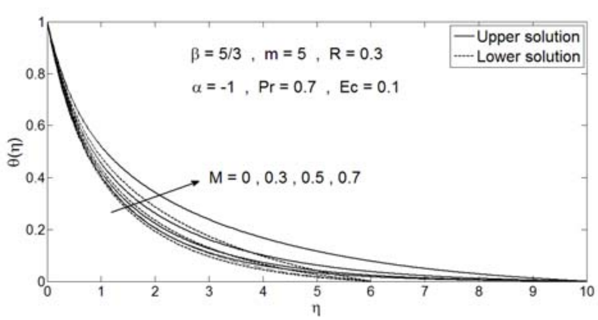

Fig. 13 The effects of Magnetic field M on the temperature field $\theta(\eta)$. In case of shrinking sheet $\alpha = -1$ (Upper and Lower solution)

In Fig. 3 the velocity $f`(\eta)$ is plotted both for numerical and analytical solutions in case of linear stretching sheet. It is apparent that numerical solution is in a good agreement with analytical solution. Fig. 4 and 5 illustrate the change in the velocity component $f`(\eta)$ in case of shrinking sheet for different values of power index m. Fig. 4 uses the upper solution while lower solution is used in Fig. 5. Fig. 4 reveals that the velocity $f`(\eta)$ decreases as we increase the values of m. However, this decrement in the velocity is smaller in case of hydromagnetic fluid when compared with hydrodynamic fluid. This is because the magnetic force acts as a resistance to the flow. It is also seen that the boundary layer thickness decreases for a smaller values of m. It is observed from Fig. 5 that near the sheet the magnitude of velocity $f`(\eta)$ increases for large values of m, while the boundary layer thickness decreases by increasing the values of m. Fig. 6 depicts the effects of the magnetic filed M on the velocity $f`(\eta)$ in case of shrinking sheet. As expected, the magnitude of velocity and boundary layer thickness decrease by increasing the values of M. Here it is found that the dual solutions occur when M = 0.1, 0.5 and 1 at $\beta = 1.5$ and s = 3.5.

Fig. 7 elucidates the change in the velocity $f`(\eta)$ for different values of mass suction parameter s. As the mass suction parameter s increases the velocity $f`(\eta)$ overshot near the shrinking sheet. Fig. 8 presents the influences of mass suction parameter s on the temperature field $\theta(\eta)$ in case of shrinking sheet. The temperature $\theta(\eta)$ decreases by increasing mass suction parameter s. The thermal boundary layer thickness also decreases as the mass suction parameter s increases.

Fig. 9 shows the change in temperature $\theta(\eta)$ for different values of Prandtl number Pr. It is evident from this figure that both the temperature and thermal boundary layer thickness decrease by increasing the values of Pr. Figs. 10 show the existence of dual solutions for the thermal boundary layer $\theta(\eta)$ in case of shrinking sheet.

Fig. 11 shows the effect of the thermal radiation on the temperature $\theta(\eta)$ of the fluid flow. It is observed that the temperature of the fluid flow increases as the thermal radiation parameter increases. It is found from Fig. 12 that the dual solutions occur when R = 0, 1, 3 and 5 at $\beta = 5/3$ and Pr = 0.7. It is further noted from Fig. 13 that the thermal boundary layer increases by increasing the values of magnetic field M.

6. CONCLUSIONS

An investigation is performed for the effects of suction/blowing and thermal radiation on a hydromagnetic viscous fluid over a non-linear stretching and shrinking sheet. The Homotopy analysis method and the numerical solution used to solve the governing equations. The following conclusions are obtained:

- The velocity increases as the power index m (Lower solution case) and mass suction parameter s increase.
- The velocity decreases as the power index m (Upper solution case) and magnetic field M increase.
- The temperature increases as the thermal radiation R and magnetic field M increase.
- The temperature decreases as the mass suction s and Prandtl number Pr increase.

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