5.4 Hydrodynamically Fully Developed and Thermally Developing Laminar Flow

- In the previous section we considered problems where the velocity and temperature profile were fully developed, so that the heat transfer coefficient was constant with distance along the pipe.
- In this section we consider problems in which the velocity is fully developed at the point where the heat transfer starts.
- Furthermore, as before, we consider two cases of constant wall temperature and wall heat flux, both by assuming uniform temperature at the inlet.
- Under these conditions the heat transfer coefficient is not constant and varies along the tube.
Whiteman and Drake (1980), Lyche and Bird (1956) and Blackwell (1985) studied the case of fully developed flow with thermal entry effects for non-Newtonian fluids. Sellars et al. (1959) obtained thermal entry length solutions for the case of a Newtonian fluid with constant wall temperature and fully developed flow, which are presented below.
We make the following assumptions in order to obtain a closed form solution for heat transfer analysis in a circular tube.

1. Incompressible Newtonian fluid
2. Laminar flow
3. Two-dimensional steady state
4. Axial conduction and viscous dissipation is neglected
5. Constant properties

This does not mean that one cannot obtain analytical solutions when one or more of the above assumptions is valid, but the solution will be much easier by making the above assumptions.
Since the fully developed velocity was already obtained in Section 5.2, we will focus the emphasis on the solution of the energy equation and boundary conditions for a developing temperature profile.
Constant Wall Temperature

The describing dimensionless energy equation (5.33) and boundary conditions with making the assumptions noted above for the case of constant wall temperature are

\[
\frac{u^+}{2} \frac{\partial \theta}{\partial x^+} = \frac{1}{r^+} \left[ \frac{\partial}{\partial r^+} \left( r^+ \frac{\partial \theta}{\partial r^+} \right) \right]
\]

(5.62)

\[\theta(r^+, 0) = 1\]
\[\theta(1, x^+) = 0\]
\[\theta(0, x^+) = \text{finite or } \frac{\partial \theta}{\partial r^+}(0, x^+) = 0\]  

(5.63)

where

\[r^+ = \frac{r}{r_o}, \quad \theta = \frac{T - T_w}{T_{in} - T_w}, \quad u^+ = \frac{u}{u_m}, \quad x^+ = \frac{x}{r_o} \frac{\text{Re Pr}}{\text{Re Pr}}\]
For a fully developed laminar flow, the parabolic velocity profile developed before is applicable.

\[ u = 2u_m \left( 1 - \frac{r^2}{r_o^2} \right) \quad \text{or} \quad u^+ = 2 \left( 1 - r^{+2} \right) \]

Substituting the above equation into the energy equation (5.62), we get

\[ \left( 1 - r^{+2} \right) \frac{\partial \theta}{\partial x^+} = \frac{\partial^2 \theta}{\partial r^{+2}} + \frac{1}{r^+} \frac{\partial \theta}{\partial r^+} \]  

(5.64)

Since the above partial differential equation is linear and homogeneous, one can apply the method of separation of variables. The separation of variables solution is assumed of the form

\[ \theta \left( r^+, x^+ \right) = R \left( r^+ \right) X \left( x^+ \right) \]  

(5.65)
The substitution of the above equation into equation (5.64) yields two ordinary differential equations

\[ X' + \lambda^2 X = 0 \] (5.66)

\[ R'' + \frac{1}{r^+} R' + \lambda^2 R \left(1 - r'^2\right) = 0 \] (5.67)

where \( \lambda^2 \) is the separation constant or eigenvalue.

The solution for equation (5.66) is a simple exponential function of the form

\[ e^{-\lambda^2 x^+} \]

while the solution of equation (5.67) is of infinite series referred to by Sturm-Liouville theory.
The final solution can be of the form

\[
\theta(r^+, x^+) = \sum_{n=0}^{\infty} c_n R_n(r^+) \exp\left(-\lambda_n^2 x^+\right) \tag{5.68}
\]

where \(\lambda_n\) are eigenvalues and \(R_n\) are eigenfunctions corresponding to equation (5.67) and \(c_n\) are constants.
The local heat flux, dimensionless mean temperature, local Nusselt number and mean Nusselt number can be obtained from the following equations, using the temperature distribution above.

\[
q_w'' = -k \left. \frac{\partial T}{\partial r} \right|_{r=r_o} = -k \left. \frac{(T_w - T_{in})}{r_o} \frac{\partial \theta}{\partial r^+} \right|_{r^+=1}
\]

\[
= -\frac{2k}{r_o} (T_w - T_{in}) \sum_{n=0}^{\infty} G_n \exp(-\lambda_n^2 x^+)
\]

\[
\theta_m = \frac{T_m - T_w}{T_{in} - T_w} = 8 \sum_{n=0}^{\infty} G_n \left[ \frac{\exp(-\lambda_n^2 x^+)}{\lambda_n^2} \right]
\]

\[
Nu_x = \frac{h_x (2r_o)}{k} = \frac{-q_w'' (2r_o)}{(T_w - T_{in}) k \theta_m} = -2 \left. \frac{\partial \theta}{\partial r^+} \right|_{r^+=1} = \sum_{n=0}^{\infty} G_n \exp(-\lambda_n^2 x^+) \]

\[
Nu_m = \frac{\bar{h}_x (2r_o)}{k} = \frac{1}{x^+} \int_0^{x^+} Nu_x \, dx^+ = -\frac{1}{2x^+} \ln \left[ \sum_{n=0}^{\infty} \frac{G_n \exp(-\lambda_n^2 x^+)}{\lambda_n^2} \right]
\]

where \( G_n = -\frac{1}{2} c_n R'_n (1) \)
The first five terms in equations (5.69) and (5.72) are fully sufficient to provide accurate solutions to the above infinite series. The eigenvalues, \( \lambda_n \) and \( G_n \), to calculate \( q_w'' \), \( \theta_m \), \( Nu_x \) and \( Nu_m \) for the above problem are presented in Table 5.3. Table 5.4 provides the variation of \( Nu_x \), \( Nu_m \) and \( \theta_m \) with distance along the tube.

It can be easily observed from Table 5.3 that the fully developed temperature profile starts at approximately

\[
x^+ = \frac{x}{r_0} = 0.1
\]

(5.73)

Therefore,

\[
(L_{T,T} / D) = 0.05RePr
\]

where \( L_{T,T} \) is the thermal entrance length for constant wall temperature.

The thermal entry length increases as the Reynolds number and Prandtl number increase. A very long thermal entry length is needed for fluids with a high Prandtl number, such as oil. Therefore, care should be taken to make a fully developed temperature profile assumption for fluids with a high Prandtl number.
Table 5.3  Eigenvalues and Eigenfunctions of a Circular Duct; Thermal Entry Effect with Fully Developed Laminar Flow and Constant Wall Temperature (Blackwell 1985)

<table>
<thead>
<tr>
<th>n</th>
<th>$\lambda_n^2/2$</th>
<th>$G_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3.656</td>
<td>0.749</td>
</tr>
<tr>
<td>1</td>
<td>22.31</td>
<td>0.544</td>
</tr>
<tr>
<td>2</td>
<td>56.9</td>
<td>0.463</td>
</tr>
<tr>
<td>3</td>
<td>107.6</td>
<td>0.414</td>
</tr>
<tr>
<td>4</td>
<td>174.25</td>
<td>0.382</td>
</tr>
</tbody>
</table>
### Table 5.4 Nusselt Solution for Thermal Entry Effect of a Circular Tube for Fully Developed Laminar Flow and Constant Wall Temperature (Blackwell 1985)

<table>
<thead>
<tr>
<th>$x^+$</th>
<th>$\text{Nu}_x$</th>
<th>$\text{Nu}_m$</th>
<th>$\theta_m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\infty$</td>
<td>$\infty$</td>
<td>1</td>
</tr>
<tr>
<td>0.0005</td>
<td>10.1</td>
<td>15.4</td>
<td>0.940</td>
</tr>
<tr>
<td>0.002</td>
<td>8.06</td>
<td>12.2</td>
<td>0.907</td>
</tr>
<tr>
<td>0.005</td>
<td>6.00</td>
<td>8.94</td>
<td>0.836</td>
</tr>
<tr>
<td>0.02</td>
<td>4.17</td>
<td>5.82</td>
<td>0.628</td>
</tr>
<tr>
<td>0.04</td>
<td>3.79</td>
<td>4.89</td>
<td>0.457</td>
</tr>
<tr>
<td>0.05</td>
<td>3.71</td>
<td>4.64</td>
<td>0.395</td>
</tr>
<tr>
<td>0.1</td>
<td>3.658</td>
<td>4.16</td>
<td>0.190</td>
</tr>
<tr>
<td>$\infty$</td>
<td>3.657</td>
<td>3.657</td>
<td>0</td>
</tr>
</tbody>
</table>
Constant Heat Flux at the Wall

- The laminar fully developed thermal entry length (developing temperature profile) for constant wall heat flux is very similar to constant wall temperature, except that dimensionless temperature and boundary conditions are defined as follows

\[
\theta = \frac{T_{in} - T}{q_w'' D / k} \quad (5.74)
\]

- At

\[
q_w'' = -k \frac{\partial T}{\partial r} \bigg|_{r=r_o} = \text{constant} \quad (5.75)
\]
Siegel et al. (1958) solved the above problem for laminar fully developed flow using separation of variables and the Strum-Liouville theory of which the result is presented below

\[
\theta = \theta^* \left( r^+, x^+ \right) + 4x^+ + r^{+2} - \frac{r^{+2}}{4} - \frac{7}{24} \tag{5.76}
\]

\[
\theta^* \left( r^+, x^+ \right) = \sum_{n=1}^{\infty} c_n R_n \exp \left( -\lambda_n^2 x^+ \right) \tag{5.77}
\]

1. The eigenvalues \( \lambda_n \), eigenfunctions \( R_n \) and constants \( c_n \) are presented in Table 5.5
Table 5.5 Eigenvalues and Eigenfunctions for Thermal Entry Effect of a Circular Tube for Fully Developed Laminar Flow and Constant Wall Heat Flux (Siegel et al. 1958)

<table>
<thead>
<tr>
<th>n</th>
<th>$\lambda_n^2$</th>
<th>$R_n(1)$</th>
<th>$C_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>25.6796</td>
<td>-0.492517</td>
<td>0.403483</td>
</tr>
<tr>
<td>2</td>
<td>83.8618</td>
<td>0.395508</td>
<td>-0.175111</td>
</tr>
<tr>
<td>3</td>
<td>174.167</td>
<td>-0.345872</td>
<td>0.105594</td>
</tr>
<tr>
<td>4</td>
<td>296.536</td>
<td>0.314047</td>
<td>-0.0732804</td>
</tr>
<tr>
<td>5</td>
<td>450.947</td>
<td>-0.291252</td>
<td>0.0550357</td>
</tr>
<tr>
<td>6</td>
<td>637.387</td>
<td>0.273808</td>
<td>-0.043483</td>
</tr>
<tr>
<td>7</td>
<td>855.850</td>
<td>-0.259852</td>
<td>0.035597</td>
</tr>
</tbody>
</table>
The local Nusselt number, based on the above solution, is given below and numerical values are presented in Table 5.6

\[ Nu_x = \left( \frac{48}{11} \right) \]

\[ 1 + \left( \frac{24}{11} \right) \sum_{n=0}^{\infty} c_n \exp \left( -\lambda_n^2 x^+ \right) R_n (1) \]

The thermal entrance for constant wall heat flux based on the numerical results presented in Table 5.6 is

\[ x^+ \approx 0.05 \]

or

\[ \left( L_{T,H} \right) = 0.05 (\text{Re Pr})(D) \] (5.79)

where \( L_{T,H} \) is the thermal entry length for fully developed flow with constant wall heat flux.
Table 5.6 Nusselt Number for Thermal Entry Effect with Fully Developed Flow of a Circular Tube with Constant Wall Heat Flux (Siegel et al. 1958)

<table>
<thead>
<tr>
<th>$x^+$</th>
<th>$Nu_x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\infty$</td>
</tr>
<tr>
<td>0.0013</td>
<td>11.5</td>
</tr>
<tr>
<td>0.0025</td>
<td>9.0</td>
</tr>
<tr>
<td>0.005</td>
<td>7.5</td>
</tr>
<tr>
<td>0.01</td>
<td>6.1</td>
</tr>
<tr>
<td>0.025</td>
<td>5.0</td>
</tr>
<tr>
<td>0.05</td>
<td>4.5</td>
</tr>
<tr>
<td>0.1</td>
<td>4.364</td>
</tr>
<tr>
<td>$\infty$</td>
<td>4.364</td>
</tr>
</tbody>
</table>
The mean temperature variation can be obtained from Nusselt number, equation (5.78), by using the following equation:

\[
T_w - T_m = \frac{q''_w}{h_x} = \frac{q''_w D}{N u_x k}
\]  

(6.80)