4.7 Integral Method

- As noted before, the similarity solution provides an analytical exact solution for laminar boundary layer conservation equations. However, there are limitations in terms of geometry and boundary conditions as well as laminar flow restrictions.

- Integral methods are approximate closed form solutions which have much less limitations in terms of geometry and boundary condition. It can also apply to both laminar and turbulent flow situations. The integral method easily provides accurate answers (not exact) for a complex problem.
In integral methods, one usually integrates the conservative differential boundary layer equation over the boundary layer thickness by assuming a profile for velocity, temperature, and concentration, as needed.

The better approximate shape for the profile, such as velocity and temperature, is the better prediction for drag force and heat transfer (friction coefficient or heat transfer coefficient).
Integral methodology has applied to a variety of configurations to solve transport phenomenon problems (Schlichting and Gersteu, 2000).

To illustrate the integral methodology, it is applied for flow and heat transfer over a wedge with non-uniform temperature and blowing at the wall.
Consider two dimensional laminar steady flow with constant properties over a wedge, as shown in figure 4.16, with an unheated starting length, \( x_0 \).

**Figure 4.16** Momentum and heat transfer over a wedge with an unheated starting length.
The governing boundary layer equation for mass, momentum and energy for constant property, steady state and laminar flow including boundary conditions for convective heat transfer over a wedge are presented below:

- **Continuity equation**
  \[
  \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{4.151}
  \]

- **Momentum equation**
  \[
  u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} + U \frac{dU}{dx} \tag{4.152}
  \]

- **Energy equation**
  \[
  u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} \tag{4.153}
  \]

- **Boundary conditions**
  \[
  u(0, x) = 0 \\
  v(0, x) = v_\infty \\
  u(\delta, x) = U(x) \\
  T(\infty, x) = T_\infty \\
  T(0, x) = T_w \quad \text{for } x < X_0 \\
  T(0, x) = T_w \quad \text{for } x > X_0
  \tag{4.154}
  \]
It should be noted that $U$ is known from potential flow theory \(-\frac{1}{\rho} \frac{\partial p}{\partial x} = U \frac{dU}{dx}\). If $U$ is constant, then

$$\frac{dU}{dx} = 0$$  \hspace{1cm} (4.155)

As for the case of flow over a flat plate, if $x_0 = 0$, $U = \text{constant}$ and $\nu_{\infty} = 0$, the problem will be similar to case presented by similarity solution before.
In integral methods, it is customary to assume a profile for \( u \) and obtain \( v \) from continuity, equation (4.151).

Let us integrate equation (4.151) with respect to \( y \) from \( y = 0 \) to \( y = \delta \). The velocity and temperature field outside \( \delta \) is uniform.

\[
\int_0^\delta \frac{\partial u}{\partial x} \, dy + \int_0^\delta \frac{\partial v}{\partial y} \, dy = 0 \tag{4.156}
\]

The second term can be easily integrated

\[
\int_0^\delta \frac{\partial v}{\partial y} \, dy = v\bigg|_{y=\delta} - v\bigg|_{y=0} = v_\delta - v_w \tag{4.157}
\]
Combining (4.156) and (4.157)

\[ \int_{0}^{\delta} \frac{\partial u}{\partial x} \, dy = v_w - v_{\delta} \]  

(4.158)

Applying Leibnitz’s formula to the left hand side of (4.158) yields

\[ \frac{\partial}{\partial x} \int_{0}^{\delta} u \, dy - u(x, \delta) \frac{d\delta}{dx} = v_w - v_{\delta} \]  

(4.159)

Or rearrange and using \( u(x, \delta) = U \)

\[ v_{\delta} = v_w + U \frac{d\delta}{dx} - \frac{\partial}{\partial x} \left( \int_{0}^{\delta} u \, dy \right) \]  

(4.160)
Let us rearrange the momentum equation, equation (4.152):

\[ \frac{\partial u^2}{\partial x} + \frac{\partial vu}{\partial y} - u \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = U \frac{dU}{dx} + \nu \frac{\partial^2 u}{\partial y^2} \]  

(4.161)

The last term is in parenthesis of the right hand side of the above equation is zero because of continuity equation. Upon integration of equation (4.161) from \( y = 0 \) to \( y = \delta \):

\[ \int_0^\delta \frac{\partial u^2}{\partial x} \, dy + \int_0^\delta \frac{\partial vu}{\partial y} \, dy = \int_0^\delta U \frac{dU}{dx} \, dy + \nu \int_0^\delta \frac{\partial^2 u}{\partial y^2} \, dy \]  

(4.162)
Upon further integration and simplification the above equation reduces to

\[
\int_0^\delta \frac{\partial u^2}{\partial x} dy + vu\bigg|_\delta - vu\bigg|_0 = \int_0^\delta U \frac{dU}{dx} dy + v \frac{\partial u}{\partial y}\bigg|_\delta - v \frac{\partial u}{\partial y}\bigg|_0
\]  (4.163)

Using equation (4.162) for \( \nu_\delta \), no slip boundary condition at wall \( u(0, x) = 0 \), and assuming no velocity gradient at the outer edge of boundary layer at \( y = \delta \) based on physical ground

\[
\int_0^\delta \frac{\partial u^2}{\partial x} dy + Uv_w + U^2 \frac{d\delta}{dx} - U \frac{\partial}{\partial x} \left( \int_0^\delta u dy \right) = -\frac{\tau_w}{\rho} + \int_0^\delta U \frac{dU}{dx} dy
\]  (4.164)

where

\[
\tau_w = -\mu \frac{\partial u}{\partial y}\bigg|_0
\]  (4.165)

is the shear stress at the wall.
4.7 Integral Method

- Apply Leibnitz's rule and rearrangement will provide the final form.

\[
\frac{\partial}{\partial x}\left[\int_0^\delta u(u-U)\,dy\right] + \left(\int_0^\delta u\,dy\right)\frac{dU}{dx} - \int_0^\delta U\frac{dU}{dx}\,dy = -\frac{\tau_w + \rho U v_w}{\rho}
\]  \hspace{1cm} (4.166)

- The only dependent unknown variable in the above equation is \( u \) since \( u \) is eliminated using continuity. \( U, \tau_w \) and \( v_w \) should be known quantities.

- Equation (4.166) can be further rearranged

\[
\frac{\partial}{\partial x}\left[U^2\int_0^\delta \frac{u}{U}\left(1-\frac{u}{U}\right)\,dy\right] + \left[\int_0^\delta \left(1-\frac{u}{U}\right)\,dy\right]U\frac{dU}{dx} = \frac{\tau_w + \rho U v_w}{\rho}
\]  \hspace{1cm} (4.167)
It is customary that assume a third order polynomial equation is for the velocity profile to obtain a reasonable result.

$$u = c_1 + c_2 y + c_3 y^2 + c_4 y^3$$  \hspace{1cm} (4.168)

$c_1$, $c_2$, $c_3$ and $c_4$ are constant and can be obtained from boundary conditions for velocity and shear stress at the wall and outer edge. Once the constants are obtained, they are substituted in the momentum integral equation (4.167) and solved for momentum boundary layer thickness, $\delta$. 
Similarly, the energy equation (4.153) can be rearranged into the following form to make the integration process easier.

\[
\frac{\partial u T}{\partial x} + \frac{\partial v T}{\partial y} - T \left( \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \right) = \alpha \frac{\partial^2 T}{\partial y^2}
\]  

Let’s integrate the above equation from \( y = 0 \) to \( y = \delta_T \), knowing the last term in parenthesis on the right hand side is zero because of continuity equation.

\[
\int_0^{\delta_T} \frac{\partial u T}{\partial x} \, dy + \int_0^{\delta_T} \frac{\partial v T}{\partial y} \, dy = \alpha \int_0^{\delta_T} \frac{\partial^2 T}{\partial y^2} \, dy
\]
Upon integration and using continuity equation to obtain $\nu_\delta$ equation (4.160) and assuming no temperature gradient at the outer edge of the thermal boundary layer, we get the following equation

$$\int_0^{\delta_T} \frac{\partial}{\partial x} (uT) dy - T_\infty \frac{\partial}{\partial x} \int_0^{\delta_T} u dy + T_\infty v_w - v_w T_w + T_\infty u_\delta \frac{d\delta_T}{dx} = \frac{-k}{\rho c_p} \frac{\partial T}{\partial y} \bigg|_{y=0} \quad (4.171)$$

Upon using Leibnitz rule and rearrangement

$$\frac{\partial}{\partial x} \left[ \int_0^{\delta_T} u (T - T_\infty) dy \right] + \left( \int_0^{\delta_T} u dy \right) \frac{dT_\infty}{dx} = \frac{q''_w}{\rho c_p} + v_w (T_w - T_\infty) \quad (4.172)$$
Once again the integral form of energy equation is in terms of the unknowns temperature assuming the velocity profile is known.

Similar to momentum integral equation, a temperature profile should be assumed and substituted in integral energy equation (4.172) to obtain $\delta_T$.

To illustrate the procedure, we use the above approximation to solve the classical problem of flow and heat transfer over a flat plate when $U = U_{\infty}$ = constant, no blowing or suction at the wall and constant wall and flow stream temperature.

Momentum and energy integral equations (4.167) and (4.172) will reduce to the following form using the above assumptions.
Let’s assume the polynomial velocity profile is third degree with the following boundary conditions.

\[ u(0) = 0 \]  \hspace{1cm} (4.175)

\[ u(\delta) = U_\infty \]  \hspace{1cm} (4.176)

\[ \left. \frac{\partial u}{\partial y} \right|_{y=\delta} = 0 \]  \hspace{1cm} (4.177)

\[ \left. \frac{\partial^2 u}{\partial y^2} \right|_{y=\delta} = 0 \]  \hspace{1cm} (4.178)

\[
\frac{\partial}{\partial x} \left[ U^2 \int_0^\delta \frac{u}{U} \left(1 - \frac{u}{U}\right) dy \right] = \frac{\tau_w}{\rho} \]

\[ \frac{\partial}{\partial x} \left[ \int_0^{\delta_x} u (T - T_\infty) dy \right] = \frac{q_w}{\rho c_p} \]  \hspace{1cm} (4.174)
It is assumed that shear stress at boundary layer edge is zero, which is a good approximation for this configuration. Equation (4.178) is obtained by using equation (4.177) and applying the x-direction momentum equation at the boundary layer edge.

Upon applying equations (4.175) through (4.178) in equation (4.168), one obtains four equations and four unknowns ($c_1$, $c_2$, $c_3$ and $c_4$). The final velocity profile is

$$\frac{u}{U} = \frac{3}{2} \left( \frac{y}{\delta} \right) - \frac{1}{2} \left( \frac{y}{\delta} \right)^3 \quad , \quad y \leq \delta$$

(4.179)
Shear stress at the wall $\tau_\omega$ is calculated using equation (4.179)

$$\tau_\omega = -\mu \frac{\partial u}{\partial y} \bigg|_{y=0} = \frac{3\mu U_\infty}{2\delta} \tag{4.180}$$

Substituting equations (4.179) and (4.180) in (4.173), and performing the integration we get

$$\frac{d}{dx} \left( \frac{39U_\infty^2 \delta}{280} \right) = \frac{3\nu U_\infty}{2\delta} \tag{4.181}$$
Integrating the above equation and assuming \( \delta = 0 \) at \( x = 0 \), we get
\[
\delta = \left( \frac{280\nu x}{13U_\infty} \right)^{1/2}
\] (4.182)

or
\[
\frac{\delta}{x} = \frac{4.64}{\text{Re}^{1/2}}
\] (4.183)

The friction coefficient is found as before
\[
\frac{c_f}{2} = \frac{\tau_w}{\rho \frac{U_\infty^2}{2}} = \frac{\mu}{\rho \frac{U_\infty^2}{2}} \frac{\partial u}{\partial y}\bigg|_{y=0}
\] (4.184)

or using equation (4.180) and (4.182)
\[
\frac{c_f}{2} = \frac{0.323}{\text{Re}^{1/2}}
\] (4.185)
The prediction of the momentum boundary layer thickness and friction coefficient $c_f$ by integral method are 7% and 3% lower than the exact solution obtained using the similarity method respectively.

Use the same general third order polynomial equation for a temperature profile with the following boundary conditions:

\begin{align}
T(0) &= T_w = \text{constant} \\
T(\delta_T) &= T_\infty = \text{constant} \\
\frac{\partial T}{\partial y}\bigg|_{y=\delta_T} &= 0 \\
\frac{\partial^2 T}{\partial y^2}\bigg|_{y=\delta_T} &= 0
\end{align}
Upon using equations (4.186) through (4.189), to find constants for the temperature profile, we get

\[
\frac{T - T_\infty}{T_w - T_\infty} = 1 - \frac{3}{2} \frac{y}{\delta_T} + \frac{1}{2} \left( \frac{y}{\delta_T} \right)^3
\]  (4.190)

\[
q_w = -k \frac{\partial T}{\partial y} \bigg|_{y=0} = \frac{3}{2} \frac{k}{\delta_T} (T_w - T_\infty)
\]  (4.191)
Substitution of equation (4.188), (4.189) and (4.190) into equation (4.173) and approximate integration for $\frac{\delta_T}{\delta} < 1$ yields:

$$
\frac{\partial}{\partial x} \int_{0}^{\delta_T} u(T - T_\infty) dy = \frac{\partial}{\partial x} \left[ U_\infty (T_w - T_\infty) \delta_T \int_{0}^{1} \frac{u}{U_\infty} \left( \frac{T - T_\infty}{T_w - T_\infty} \right) d \left( \frac{y}{\delta_T} \right) \right]
$$

$$
= \frac{\partial}{\partial x} \left[ \frac{3}{20} \delta_T^2 \left( 1 - \frac{\delta_T^2}{14 \delta^2} \right) U_\infty (T_w - T_\infty) \right]
$$

$$
= \frac{q_w}{\rho c_p} = \frac{3}{2} \frac{\alpha}{\delta_T} (T_w - T_\infty)
$$

(4.192)
Upon further simplification
\[
\frac{d}{dx} \left[ \frac{\delta_T^2}{\delta} \left(1 - \frac{\delta_T^2}{14\delta^2} \right) \right] = \frac{10\nu}{\text{Pr} U_\infty} \frac{1}{\delta_T}
\]  
(4.193)

The only unknown in the above equation is \(\delta_T\) since \(\delta\) is known. Assume \(\zeta = \frac{\delta_T}{\delta}\).
\[
\frac{d}{dx} \left[ \zeta^2 \delta \left(1 - \frac{\zeta^2}{14} \right) \right] = \frac{10\nu}{\text{Pr} U_\infty} \frac{1}{\zeta \delta}
\]  
(4.194)

The solution of the above equation for \(\zeta=0\) at \(x=x_0\) yields
\[
\zeta = \frac{\text{Pr}^{-\frac{1}{3}}}{1.026} \left[1 - \left(\frac{x_0}{x}\right)^{\frac{3}{4}}\right]^{\frac{1}{3}}
\]  
(4.195)
The local heat transfer coefficient can now be calculated since $\delta_T$ is known.

$$h = \frac{q_w}{T_w - T_\infty} = \frac{-k \frac{\partial T}{\partial y}}{T_w - T_\infty} = \frac{3}{2} \frac{k(T_w - T_\infty)}{\delta_T}$$  \hspace{1cm} (4.196)

or

$$h = \frac{3}{2} \frac{k}{\zeta \delta_T}$$  \hspace{1cm} (4.197)

Using $\zeta$ for equation (4.195) and $\delta$ from equation (4.69) the local Nusselt number $Nu_x = \frac{hx}{k}$ is

$$Nu_x = \frac{0.332 Pr^{1/2} Re_x^{1/2}}{\left[1 - \left(\frac{x_0}{x}\right)^{3/4}\right]^{1/3}}$$  \hspace{1cm} (4.198)

The above equation in this case with no unheated starting length ($\zeta=0$) reduces to the exact solution obtained by the similarity solution.

$$Nu_x = 0.332 Pr^{1/2} Re_x^{1/2}$$  \hspace{1cm} (4.199)