3.3 Unsteady State Heat Conduction

- For many applications, it is necessary to consider the variation of temperature with time.
- In this case, the energy equation for classical heat conduction, eq. (3.8), should be solved.
- If the thermal conductivity is independent from the temperature, the energy equation is reduced to eq. (3.10).

3.3.1 Lumped Analysis

Consider an arbitrarily shaped object with volume V, surface area A_s, and a uniform initial temperature of T_i as shown in Fig. 3.15. At time t = 0, the arbitrarily shaped object is exposed to a fluid with temperature of T_∞ and the convective heat transfer coefficient between the fluid and the arbitrarily shaped object is h.

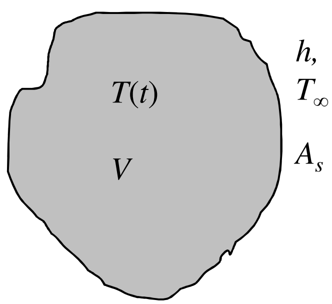

Figure 3.15 Lumped capacitance method

The convective heat transfer from the surface is

$$q = hA_s(T - T_\infty)$$

Similar to what we did for heat transfer from an extended surface, the heat loss due to surface convection can be treated as an equivalent heat source

$$q''' = -\frac{q}{V} = -\frac{hA_s(T - T_\infty)}{V}$$

Since the temperature is assumed to be uniform, the energy equation becomes

$$\rho c_p \frac{\partial T}{\partial t} = -\frac{hA_s(T - T_\infty)}{V} \qquad (3.188)$$

which is subject to the following initial condition

$$T = T_i, \quad t = 0 \qquad (3.189)$$

Introducing excess temperature, $\vartheta = T - T_\infty$, eqs. (3.188) and (3.189) become

$$\frac{d\vartheta}{dt} = -\frac{hA_s}{\rho V c_p}\vartheta \qquad (3.190)$$

$$\vartheta = \vartheta_i, \quad t = 0 \qquad (3.191)$$

Integrating eq. (3.190) and determining the integral constant using eq. (3.191), the solution becomes

$$\frac{\vartheta}{\vartheta_i} = e^{-t/\tau_t} \qquad (3.192)$$

where

$$\tau_t = \frac{\rho V c_p}{hA_s} \qquad (3.193)$$

is referred to as the thermal time constant.

The cooling process requires transferring heat from the center of the object to the surface and the heat is further transferred away from the surface by convection. When the lamped capacitance method is employed, it is assumed that the conduction resistance within the object is negligible compared with the convective thermal resistance at the surface, therefore, the validity of the lumped analysis depends on the relative thermal resistances of conduction and convection. The conduction thermal resistance can be expressed as

$$R_{cond} = \frac{L_c}{kA_c}$$

where L_c is the characteristic length and A_c is the area of heat conduction.

The convection thermal resistance at the surface is

$$R_{conv} = \frac{1}{hA_s}$$

Assuming $A_s = A_c$, one can define the Biot number, Bi, as the ratio of the conduction and convection thermal resistances

$$\mathrm{Bi} = \frac{R_{cond}}{R_{conv}} \doteq \frac{hL_c}{k} \qquad (3.194)$$

If the characteristic length is chosen as $L_c = V / A_s$, the lumped capacitance method is valid when the Biot number is less than 0.1, or the conduction thermal resistance is one order of magnitude smaller than the convection thermal resistance at the surface.

3.3.2 One Dimensional Transient Systems

For the case that the Biot number is greater than 0.1, the temperature distribution can no longer be treated as uniform and the knowledge about the temperature distribution is also of interest. We will now consider the situation where temperature only varies in one spatial dimension. Both homogeneous and nonhomogeneous problem will be considered.

Homogeneous Problems

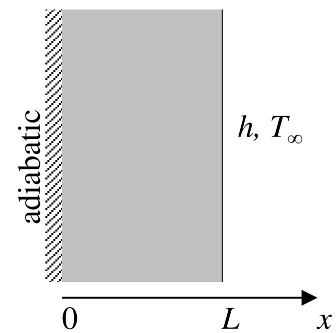

Figure 3.16 Transient conduction in a finite slab

Figure 3.16 shows a finite slab with thickness of L and a uniform initial temperature of T_i. At time t = 0, the left side of the slab is insulated while the right side of the slab is exposed to a fluid with temperature of T_∞. In contrast to the lumped capacitance method that assumes uniform temperature, we will present a more generalized model that takes non-uniform temperature distribution into account.

The energy equation for this one-dimensional transient conduction problem is

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}, \quad 0 < x < L,\ t > 0 \tag{3.195}$$

subject to the following boundary and initial conditions

$$\frac{\partial T}{\partial x} = 0, \quad x = 0 \tag{3.196}$$

$$-k\frac{\partial T}{\partial x} = h(T - T_\infty), \quad x = L \tag{3.197}$$

$$T = T_i, \quad 0 < x < L,\ t = 0 \tag{3.198}$$

This is a nonhomogeneous problem because eq. (3.197) is not homogeneous. By introducing the excess temperature, $\vartheta = T - T_\infty$, the problem can be homogenized, i.e.,

$$\frac{\partial^2 \vartheta}{\partial x^2} = \frac{1}{\alpha}\frac{\partial \vartheta}{\partial t}, \quad 0 < x < L,\ t > 0 \tag{3.199}$$

$$\frac{\partial \vartheta}{\partial x} = 0, \quad x = 0 \tag{3.200}$$

$$-k\frac{\partial \vartheta}{\partial x} = h\vartheta, \quad x = L \tag{3.201}$$

$$\vartheta = \vartheta_i = T_i - T_\infty, \quad 0 < x < L, \ t = 0 \tag{3.202}$$

To express our solution in a compact form so that it can be used for all similar problems, one can define the following dimensionless variables

$$\theta = \frac{\vartheta}{\vartheta_i}, \ X = \frac{x}{L}, \ \mathrm{Fo} = \frac{\alpha t}{L^2} \tag{3.203}$$

and eqs. (3.199) – (3.202) will be nondimensionalized as

$$\frac{\partial^2 \theta}{\partial X^2} = \frac{\partial \theta}{\partial \mathrm{Fo}}, \quad 0 < X < 1, \ \mathrm{Fo} > 0 \tag{3.204}$$

$$\frac{\partial \theta}{\partial X} = 0, \quad X = 0 \tag{3.205}$$

$$-\frac{\partial \theta}{\partial X} = \mathrm{Bi}\,\theta, \quad \mathrm{X} = 1 \tag{3.206}$$

$$\theta = 1, \quad 0 < X < 1, \ \mathrm{Fo} = 0 \tag{3.207}$$

This problem can be solved using the method of separation of variables. Assuming that the temperature can be expressed as

$$\theta(X, \mathrm{Fo}) = \Theta(X)\Gamma(\mathrm{Fo}) \tag{3.208}$$

where Θ and Γ are functions of *X* and Fo, respectively, eq. (3.204) becomes

$$\frac{\Theta''(X)}{\Theta(X)} = \frac{\Gamma'(\mathrm{Fo})}{\Gamma(\mathrm{Fo})}$$

Since the left hand side is a function of *X* only and the right-hand side of the above equation is a function of Fo only, both sides must be equal to a separation constant, , i.e.,

$$\frac{\Theta''(X)}{\Theta(X)} = \frac{\Gamma'(\mathrm{Fo})}{\Gamma(\mathrm{Fo})} = \mu \qquad (3.209)$$

The separation constant μ can be either a real or a complex number. The solution of Γ from eq. (3.209) will be $\Gamma = e^{\mu \mathrm{Fo}}$. If μ is a positive real number, we will have $\Gamma \to \infty$ when $\mathrm{Fo} \to \infty$, which does not make sense, therefore, μ cannot be a positive real number. If μ is zero, we will have $\Gamma = \mathrm{const}$, and Θ is a linear function of X only. The final solution for θ will also be a linear function of X only, which does not make sense either. It can also be shown that the separation constant cannot be a complex number, therefore, the separation variable has to be a negative real number.

If we represent this negative number by $\mu = -\lambda^2$, eqs. (3.209) can be rewritten as the following two equations

$$\Theta'' + \lambda^2 \Theta = 0 \tag{3.210}$$

$$\Gamma' + \lambda^2 \Gamma = 0 \tag{3.211}$$

The general solutions of eqs. (3.210) and (3.211) are

$$\Theta = C_1 \cos \lambda X + C_2 \sin \lambda X \tag{3.212}$$

$$\Gamma = C_3 e^{-\lambda^2 \mathrm{Fo}} \tag{3.213}$$

where C_1, C_2, and C_3 are integral constants.

Substituting eq. (3.208) into eqs. (3.205) and (3.206), the following boundary conditions of eq. (3.210) are obtained

$$\Theta'(0) = 0 \tag{3.214}$$

$$-\Theta'(1) = \mathrm{Bi}\Theta(1) \tag{3.215}$$

Substituting eq. (3.212) into eq. (3.214) yields

$$\Theta'(0) = -C_1\lambda\sin(0) + C_2\lambda\cos(0) = C_2\lambda = 0$$

Since λ cannot be zero, C_2 must be zero and eq. (3.215) becomes

$$\Theta = C_1\cos\lambda X \qquad (3.216)$$

Applying the convection boundary condition, eq. (3.215), one obtains

$$\frac{\lambda_n}{\mathrm{Bi}} = \cot\lambda_n \qquad (3.217)$$

where n is an integer.

The dimensionless temperature with eigenvalue λ_n can be obtained by substituting eqs. (3.216) and (3.213) into eq. (4.208), i.e.,

$$\theta_n = C_n\cos\left(\lambda_n X\right)e^{-\lambda_n^2\mathrm{Fo}} \qquad (3.218)$$

where $C_n = C_1 C_3$.

Since the one-dimensional transient heat conduction problem under consideration is a linear problem, the sum of different θ_n for each value of *n* also satisfies eqs. (3.204) – (3.206).

$$\theta = \sum_{n=1}^{\infty} C_n \cos(\lambda_n X) e^{-\lambda_n^2 \mathrm{Fo}} \tag{3.219}$$

Substituting eq. (3.219) into eq. (3.207) yields

$$1 = \sum_{n=1}^{\infty} C_n \cos(\lambda_n X)$$

Multiplying the above equation by $\cos(\lambda_m X)$ and integrating the resulting equation in the interval of (0, 1), one obtains

$$\int_0^1 \cos(\lambda_m X) dX = \sum_{n=1}^{\infty} C_n \int_0^1 \cos(\lambda_m X) \cos(\lambda_n X) dX \tag{3.220}$$

The integral in the right-hand side of eq. (3.133) can be evaluated as

$$\int_0^1 \cos(\lambda_m X)\cos(\lambda_n X)dX = \begin{cases} \dfrac{\lambda_m \sin\lambda_m \cos\lambda_n - \lambda_n \cos\lambda_m \sin\lambda_n}{\lambda_m^2 - \lambda_n^2}, & m \neq n \\ \dfrac{1}{2\lambda_m}\left(\lambda_m + \dfrac{\sin 2\lambda_m}{2}\right), & m = n \end{cases} \quad (3.221)$$

Equation (3.217) can be rewritten as

$$\mathrm{Bi} = \lambda_n \tan\lambda_n$$

Similarly, for eigenvalue λ_m , we have

$$\mathrm{Bi} = \lambda_m \tan\lambda_m$$

Combining the above two equations, we have

$$\lambda_m \tan\lambda_m - \lambda_n \tan\lambda_n = 0$$

or

$$\lambda_m \sin\lambda_m \cos\lambda_n - \lambda_n \cos\lambda_m \sin\lambda_n = 0$$

therefore, the integral in eq. (3.221) is zero for the case that $m \neq n$, and the right hand side of eq. (3.220) becomes

$$\sum_{n=1}^{\infty} C_n \int_0^1 \cos(\lambda_m X)\cos(\lambda_n X)dX = \frac{C_m}{2\lambda_m}\left(\lambda_m + \frac{\sin 2\lambda_m}{2}\right) \qquad (3.222)$$

Substituting eq. (3.222) into eq. (3.220) and evaluating the integral at the left-hand side of eq. (3.220), we have

$$\frac{1}{\lambda_m}\sin\lambda_m = \frac{C_m}{2\lambda_m}\left(\lambda_m + \frac{\sin 2\lambda_m}{2}\right)$$

i.e.,

$$C_m = \frac{4\sin\lambda_m}{2\lambda_m + \sin 2\lambda_m}$$

Changing notation from *m* to *n*, we get

$$C_n = \frac{4\sin\lambda_n}{2\lambda_n + \sin 2\lambda_n} \qquad (3.223)$$

The dimensionless temperature, therefore, becomes

$$\theta = \sum_{n=1}^{\infty} \frac{4\sin\lambda_n}{2\lambda_n + \sin 2\lambda_n} \cos(\lambda_n X) e^{-\lambda_n^2 \mathrm{Fo}} \qquad (3.224)$$

If the Biot number becomes infinite, the convection boundary condition becomes

$$\theta = 0, \quad \mathrm{X} = 1 \qquad (3.225)$$

which is an isothermal condition at the right-hand side of the wall. Equation (3.217) becomes

$$\cos\lambda_n = 0 \qquad (3.226)$$

and the eigenvalue is therefore

$$\lambda_n = (n - \pi/2), \; n = 1,2,3,... \qquad (3.227)$$

The temperature distribution for this case is then

$$\theta = \sum_{n=1}^{\infty} \frac{4\sin(n-\pi/2)}{2(n-\pi/2)+\sin 2(n-\pi/2)} \cos\left[(n-\pi/2)X\right] e^{-(n-\pi/2)^2 \mathrm{Fo}} \tag{3.228}$$

When Fourier's number is greater than 0.2, only the first term in eq. (3.219) is necessary and the solution becomes

$$\theta = \theta_1 = C_1 \cos\left(\lambda_1 X\right) e^{-\lambda_1^2 \mathrm{Fo}} \tag{3.229}$$

Example 3.3

A long cylinder with radius of r_o and a uniform initial temperature of T_i is exposed to a fluid with temperature of T_∞. The convective heat transfer coefficient between the fluid and cylinder is h. Assuming that there is no internal heat generation and constant thermophysical properties, obtain the transient temperature distribution in the cylinder.

- **Solution**

  Since the temperature changes along the r-direction only, the energy equation is

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} = \frac{1}{\alpha}\frac{\partial T}{\partial t}, \quad 0 < x < r_o,\ t > 0 \qquad (3.230)$$

subject to the following boundary and initial conditions

$$\frac{\partial T}{\partial r} = 0, \quad r = 0 \text{ (axisymmetric)} \qquad (3.231)$$

$$-k\frac{\partial T}{\partial r} = h(T - T_\infty), \quad r = r_o \qquad (3.232)$$

$$T = T_i, \quad 0 < r < r_o,\ t = 0 \qquad (3.233)$$

Defining the following dimensionless variables

$$\theta = \frac{T - T_\infty}{T_i - T_\infty},\ R = \frac{r}{r_o},\ \mathrm{Fo} = \frac{\alpha t}{r_o^2},\ \mathrm{Bi} = \frac{h r_o}{k} \qquad (3.234)$$

eqs. (3.230) – (4.233) will be nondimensionalized as

$$\frac{\partial^2\theta}{\partial R^2}+\frac{1}{R}\frac{\partial\theta}{\partial R}=\frac{\partial\theta}{\partial \mathrm{Fo}},\quad 0<R<1,\ \mathrm{Fo}>0 \tag{3.235}$$

$$\frac{\partial\theta}{\partial R}=0,\quad R=0 \tag{3.236}$$

$$-\frac{\partial\theta}{\partial R}=\mathrm{Bi}\,\theta,\quad \mathrm{R}=1 \tag{3.237}$$

$$\theta=1,\quad 0<R<1,\ \mathrm{Fo}=0 \tag{3.238}$$

Assuming that the temperature can be expressed as

$$\theta(R,\mathrm{Fo})=\Theta(R)\Gamma(\mathrm{Fo}) \tag{3.239}$$

and substituting eq. (3.239) into eq. (3.235), one obtains

$$\frac{1}{\Theta}\left[\Theta''+\frac{1}{R}\Theta'\right]=\frac{\Gamma'}{\Gamma}=-\lambda^2 \tag{3.240}$$

which can be rewritten as the following two equations

$$\Theta'' + \frac{1}{R}\Theta' + \lambda^2\Theta = 0 \quad (3.241)$$

$$\Gamma' + \lambda^2\Gamma = 0 \quad (3.242)$$

Equation (3.101) is a Bessel's equation of zero order and has the following general solution

$$\Theta(R) = C_1 J_0(\lambda R) + C_2 Y_0(\lambda R) \quad (3.243)$$

where J_0 and Y_0 are the Bessel's function of the first and second kind, respectively. The general solution of eq. (3.242) is

$$\Gamma = C_3 e^{-\lambda^2 \mathrm{Fo}} \quad (3.244)$$

where C_1, C_2, and C_3 are integral constants.

The boundary conditions for eq. (3.241) can be obtained by substituting eq. (3.239) into eqs. (3.236) and (3.237), i.e.,

$$\Theta'(0) = 0 \tag{3.245}$$

$$-\Theta'(1) = \mathrm{Bi}\Theta(1) \tag{3.246}$$

The derivative of Θ is

$$\Theta'(R) = -C_1\lambda J_1(\lambda R) - C_2\lambda Y_1(\lambda R) \tag{3.247}$$

Since $J_1(0) = 0$ and $Y_1(0) = -\infty$, C_2 must be zero. Substituting eqs. (3.243) and (3.247) into eq. (3.246) and considering $C_2 = 0$ we have

$$-\lambda_n J_1(\lambda_n) + \mathrm{Bi} J_0(\lambda_n) = 0 \tag{3.248}$$

where n is an integer.

The dimensionless temperature with eigenvalue λ_n is

$$\theta_n = C_n J_0(\lambda_n R) e^{-\lambda_n^2 \mathrm{Fo}} \tag{3.249}$$

where $C_n = C_1 C_3$.

For a linear problem, the sum of different θ_n for each value of *n* also satisfies eqs. (3.235) – (3.237).

$$\theta = \sum_{n=1}^{\infty} C_n J_0(\lambda_n R) e^{-\lambda_n^2 \mathrm{Fo}} \tag{3.250}$$

Substituting eq. (3.250) into eq. (3.238) yields

$$1 = \sum_{n=1}^{\infty} C_n J_0(\lambda_n R)$$

Multiplying the above equation by $RJ_0(\lambda_m R)$ and integrating the resulting equation in the interval of (0, 1), one obtains

$$\int_0^1 RJ_0(\lambda_m R)dR = \sum_{n=1}^{\infty} C_n \int_0^1 RJ_0(\lambda_m R)J_0(\lambda_n R)dR$$

According to the orthogonal property of Bessel's function, the integral on the right-hand side equals zero if $m \neq n$ but it is not zero if *m* = *n* Therefore, we have

$$C_m = \frac{\int_0^1 R J_0(\lambda_m R) dR}{\int_0^1 R J_0^2(\lambda_m R) dR} = \frac{2}{\lambda_m} \frac{J_1(\lambda_m)}{J_0^2(\lambda_m) + J_1^2(\lambda_m)}$$

Changing notation from *m* to *n*, we get

$$C_n = \frac{2}{\lambda_n} \frac{J_1(\lambda_n)}{J_0^2(\lambda_n) + J_1^2(\lambda_n)} \qquad (3.251)$$

thus, the dimensionless temperature becomes

$$\theta = \sum_{n=1}^{\infty} \frac{2}{\lambda_n} \frac{J_1(\lambda_n) J_0(\lambda_n R)}{J_0^2(\lambda_n) + J_1^2(\lambda_n)} e^{-\lambda_n^2 \mathrm{Fo}} \qquad (3.252)$$

One-dimensional heat conduction in a spherical coordinate system can be solved by introducing a new dependent variable. Consider a sphere with radius of r_o and a uniform initial temperature of T_i. It is exposed to a fluid with a temperature of T_∞ and the convective heat transfer coefficient between the fluid and finite slab is h. Assuming that there is no internal heat generation and constant thermophysical properties, the governing equation is

$$\frac{1}{r}\frac{\partial^2 (rT)}{\partial r^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}, \quad 0 < x < r_o,\ t > 0 \tag{3.253}$$

subject to the following boundary and initial conditions

$$\frac{\partial T}{\partial r} = 0, \quad r = 0 \text{ (axisymmetric)} \tag{3.254}$$

$$-k\frac{\partial T}{\partial r} = h(T - T_\infty), \quad r = r_o \tag{3.255}$$

$$T = T_i, \quad 0 < r < r_o, \; t = 0 \tag{3.256}$$

By using the same dimensionless variables defined in eq. (3.234), eqs. (3.253) – (3.256) can be nondimensionalized as

$$\frac{1}{R}\frac{\partial^2 (R\theta)}{\partial R^2} = \frac{\partial \theta}{\partial \mathrm{Fo}}, \quad 0 < R < 1, \; \mathrm{Fo} > 0 \tag{3.257}$$

$$\frac{\partial \theta}{\partial R} = 0, \quad R = 0 \tag{3.258}$$

$$-\frac{\partial \theta}{\partial R} = \mathrm{Bi}\,\theta, \quad \mathrm{R} = 1 \tag{3.259}$$

$$\theta = 1, \quad 0 < R < 1, \; \mathrm{Fo} = 0 \tag{3.260}$$

Defining a new dependent variable

$$U = R\theta \tag{3.261}$$

eqs. (3.257) – (3.260) become

$$\frac{\partial^2 U}{\partial R^2} = \frac{\partial U}{\partial \mathrm{Fo}}, \quad 0 < R < 1,\ \mathrm{Fo} > 0 \tag{3.262}$$

$$\frac{\partial \theta}{\partial R} = 0, \quad R = 0 \tag{3.263}$$

$$-\frac{\partial U}{\partial R} = (\mathrm{Bi\text{-}1})\theta, \quad \mathrm{R} = 1 \tag{3.264}$$

$$U = R, \quad 0 < R < 1,\ \mathrm{Fo} = 0 \tag{3.265}$$

This problem can be readily solved by using the method of separation of variables. After the solution is obtained, one can change the dependent variable back to θ and the result is

$$\theta = \frac{1}{R}\sum_{n=1}^{\infty} \frac{4[\sin(\lambda_n) - \lambda_n \cos(\lambda_n)]}{\lambda_n[2\lambda_n - \sin(2\lambda_n)]} \sin(\lambda_n R) e^{-\lambda_n^2 \mathrm{Fo}} \tag{3.266}$$

where the eigenvalue is the positive root of the following equation

$$1 - \lambda_n \cot \lambda_n = \mathrm{Bi} \qquad (3.267)$$

Nonhomogeneous Problems

The solution of a nonhomogeneous problem can be obtained by superposition of a particular solution of the nonhomogeneous problem and the general solution of the corresponding homogeneous problem.

Consider a finite slab with thickness of L and a uniform initial temperature of T_i as shown in Fig. 3.17.

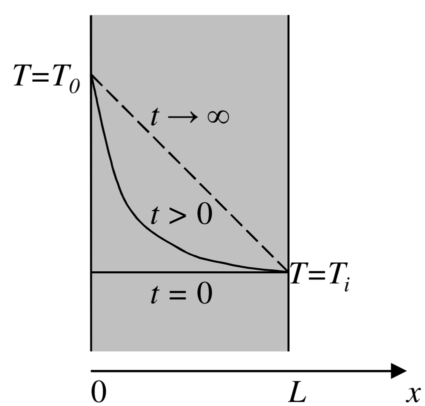

Figure 3.17 Heat conduction under boundary condition of the first kind

Assuming that there is no internal heat generation in the slab and the thermophysical properties of the slab are constants, the energy equation is

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}, \quad 0 < x < L,\ t > 0 \qquad (3.268)$$

subject to the following boundary and initial conditions

$$T = T_0, \quad x = 0 \qquad (3.269)$$

$$T = T_i, \quad x = L \qquad (3.270)$$

$$T = T_i, \quad 0 < x < L,\ t = 0 \qquad (3.271)$$

By defining the following dimensionless variables

$$\theta = \frac{T - T_i}{T_0 - T_i},\ X = \frac{x}{L},\ \mathrm{Fo} = \frac{\alpha t}{L^2} \qquad (3.272)$$

eqs. (3.268) – (3.271) will be nondimensionalized as:

$$\frac{\partial^2\theta}{\partial X^2}=\frac{\partial\theta}{\partial \mathrm{Fo}},\quad 0<X<1,\ \mathrm{Fo}>0 \tag{3.273}$$

$$\theta=1,\quad X=0 \tag{3.274}$$

$$\theta=0,\quad \mathrm{X}=1 \tag{3.275}$$

$$\theta=0,\quad 0<X<1,\ \mathrm{Fo}=0 \tag{3.276}$$

If the steady state temperature is represented by θ_s , it must satisfy the following equations:

$$\frac{\partial^2\theta_s}{\partial X^2}=0,\quad 0<X<1 \tag{3.277}$$

$$\theta_s=1,\quad X=0 \tag{3.278}$$

$$\theta_s=0,\quad \mathrm{X}=1 \tag{3.279}$$

which have the following solution:

$$\theta_s=1-X \tag{3.280}$$

To obtain the generation of the problem described by eqs. (3.273) – (3.276), a method of partial solution will be employed. In this methodology, it is assumed that the solution of a nonhomogeneous problem can be expressed as

$$\theta(X,\mathrm{Fo}) = \theta_s(X) + \theta_h(X,\mathrm{Fo}) \tag{3.281}$$

where θ_h represent the solution of a homogeneous problem. Substituting eqs. (3.273) – (3.276) and considering eqs. (3.277) – (3.279), we have

$$\frac{\partial^2\theta_h}{\partial X^2} = \frac{\partial\theta_h}{\partial \mathrm{Fo}}, \quad 0 < X < 1,\ \mathrm{Fo} > 0 \tag{3.282}$$

$$\theta_h = 0, \quad X = 0 \tag{3.283}$$

$$\theta_h = 0, \quad \mathrm{X} = 1 \tag{3.284}$$

$$\theta_h = X - 1, \quad 0 < X < 1,\ \mathrm{Fo} = 0 \tag{3.285}$$

which represent a new homogeneous problem. This problem can be solved using the method of separation of variables and the result is

$$\theta_h = -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin(n\pi X)}{n}e^{-(n\pi)^2 \mathrm{Fo}} \tag{3.286}$$

The solution of the nonhomogeneous problem thus becomes

$$\theta = 1 - X - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin(n\pi X)}{n}e^{-(n\pi)^2 \mathrm{Fo}} \tag{3.287}$$

If the steady-state solution does not exist, we can use the method of variation of parameter to solve the problem. Consider a finite slab with thickness of L and a uniform initial temperature of T_i. At time t = 0, the left side is subject to a constant heat flux while the right side of the slab is adiabatic (see Fig. 3.18).

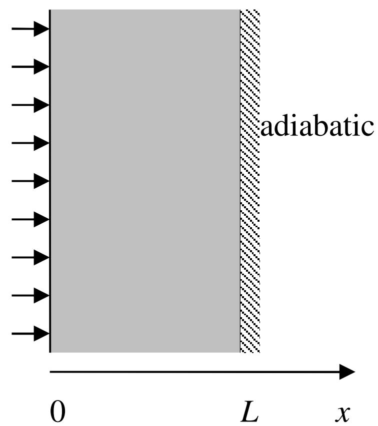

Figure 3.18 Heat conduction under boundary condition of the second kind

Assuming that there is no internal heat generation in the slab and the thermophysical properties of the slab are constants, the energy equation is

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}, \quad 0 < x < L,\ t > 0 \tag{3.288}$$

subject to the following boundary and initial conditions

$$-k\frac{\partial T}{\partial x} = q_0'', \quad x = 0 \tag{3.289}$$

$$\frac{\partial T}{\partial x} = 0, \quad x = L \tag{3.290}$$

$$T = T_i, \quad 0 < x < L,\ t = 0 \tag{3.291}$$

By defining the following dimensionless variables

$$\theta = \frac{T - T_i}{q_0'' L / k},\ X = \frac{x}{L},\ \mathrm{Fo} = \frac{\alpha t}{L^2} \tag{3.292}$$

eqs. (3.268) – (3.271) will be nondimensionalized as:

$$\frac{\partial^2\theta}{\partial X^2} = \frac{\partial\theta}{\partial \mathrm{Fo}}, \quad 0 < X < 1,\ \mathrm{Fo} > 0 \tag{3.293}$$

$$\frac{\partial\theta}{\partial X} = -1, \quad X = 0 \tag{3.294}$$

$$\frac{\partial\theta}{\partial X} = 0, \quad X = 1 \tag{3.295}$$

$$\theta = 0, \quad 0 < X < 1,\ \mathrm{Fo} = 0 \tag{3.296}$$

We will use the method of variation of parameters to solve this problem. This method requires the following steps:

1. Set up a homogeneous problem by dropping the nonhomogeneous terms,
2. Solve the homogeneous problem to get eigenvalue λ_n and eigenfunctions $\Theta_n(X)$
3. Assuming the solution of the original nonhomogeneous problem has the format of $\theta(X,\mathrm{Fo}) = \sum_{n=1}^{n} A_n(\mathrm{Fo})\Theta_n(X)$

4. Solve for A_n(Fo) using orthogonal property of Θ_n
5. Obtain an ordinary differential equation (ODE) for A_n (Fo) and solve for A_n (Fo) from the ODE
6. Put together the final solution.

We will solve this nonhomogeneous problem be following the above procedure. The corresponding homogeneous problem is:

$$\frac{\partial^2 \theta_h}{\partial X^2} = \frac{\partial \theta_h}{\partial \mathrm{Fo}}, \quad 0 < X < 1,\ \mathrm{Fo} > 0 \tag{3.297}$$

$$\frac{\partial \theta_h}{\partial X} = 0, \quad X = 0 \tag{3.298}$$

$$\frac{\partial \theta_h}{\partial X} = 0, \quad X = 1 \tag{3.299}$$

$$\theta_h = 0, \quad 0 < X < 1,\ \mathrm{Fo} = 0 \tag{3.300}$$

Assuming the solution of the above homogeneous problem is

$$\theta_h = \Theta(X)\Gamma(\mathrm{Fo}) \quad (3.301)$$

eq. (3.297) becomes

$$\frac{\Theta''(X)}{\Theta(X)} = \frac{\Gamma'(\mathrm{Fo})}{\Gamma(\mathrm{Fo})} = -\lambda^2 \quad (3.302)$$

The eigenvalue problem is

$$\Theta'' + \lambda^2\Theta = 0 \quad (3.303)$$

$$\Theta'(0) = 0 \quad (3.304)$$

$$\Theta'(1) = 0 \quad (3.305)$$

Solving eqs. (3.303) – (3.305) yields the following eigenvalues and eigen functions

$$\lambda_n = n\pi \quad (3.306)$$

$$\Theta_n(X) = \cos(n\pi X), \quad n = 0,1,2,... \quad (3.307)$$

Now, let us assume that the solution of the original nonhomogeneous problem is

$$\theta(X,\text{Fo}) = \sum_{n=0}^{\infty} A_n(\text{Fo})\cos(n\pi X) \tag{3.308}$$

Multiplying eq. (3.308) by $\cos(m\pi X)$ and integrating the resulting equation in the interval of (0, 1), one obtains

$$\int_0^1 \theta(X,\text{Fo})\cos(m\pi X)dX = \sum_{n=1}^{\infty} A_n \int_0^1 \cos(m\pi X)\cos(n\pi X)dX \tag{3.309}$$

The integral on the right-hand side of eq. (3.309) can be evaluated as

$$\int_0^1 \cos(m\pi X)\cos(n\pi X)dX = \begin{cases} 0, & m \neq n \\ 1/2, & m = n \neq 0 \\ 1, & m = n = 0 \end{cases} \tag{3.310}$$

thus, eq. (3.309) becomes

$$A_0(\text{Fo}) = \int_0^1 \theta(X,\text{Fo})dX, \qquad m = 0 \tag{3.311}$$

$$A_m(\mathrm{Fo}) = 2\int_0^1 \theta(X,\mathrm{Fo})\cos(m\pi X)dX,\ m \neq 0 \qquad (3.312)$$

Differentiating eq. (3.311) with respect to Fo, one obtains:

$$\frac{dA_0}{d\mathrm{Fo}} = \int_0^1 \frac{\partial \theta}{\partial \mathrm{Fo}} dX \qquad (3.313)$$

Substituting eq. (3.293) into eq. (3.313) and integrating with respect to *X* yield

$$\frac{dA_0}{d\mathrm{Fo}} = \int_0^1 \frac{\partial^2 \theta}{\partial X^2} dX = \left.\frac{\partial \theta}{\partial X}\right|_{X=1} - \left.\frac{\partial \theta}{\partial X}\right|_{X=0} = 1 \qquad (3.314)$$

Integrating eq. (3.314) with respect to Fo, we have

$$A_0(\mathrm{Fo}) = \mathrm{Fo} + C_1 = \int_0^1 \theta(X,\mathrm{Fo})dX \qquad (3.315)$$

When Fo = 0, eq. (3.315) becomes

$$A_0(0) = C_1 = \int_0^1 \theta(X,0)dX = 0 \qquad (3.316)$$

thus, we have

$$A_0(\mathrm{Fo}) = \mathrm{Fo} \tag{3.317}$$

Differentiating eq. (3.312) and considering eq. (3.293) yield

$$\frac{dA_\mathrm{m}}{d\mathrm{Fo}} = 2\int_0^1 \frac{\partial\theta}{\partial\mathrm{Fo}}\cos(m\pi X)dX = 2\int_0^1 \frac{\partial^2\theta}{\partial X^2}\cos(m\pi X)dX \tag{3.318}$$

Using integration by parts twice, the following ODE is obtained:

$$\frac{dA_\mathrm{m}}{d\mathrm{Fo}} = 2-(m\pi)^2 A_m \tag{3.319}$$

Multiplying eq. (3.319) by an integrating factor $e^{(m\pi)^2\mathrm{Fo}}$, we have

$$\frac{d}{d\mathrm{Fo}}\left[A_\mathrm{m}e^{(m\pi)^2\mathrm{Fo}}\right] = 2e^{(m\pi)^2\mathrm{Fo}} \tag{3.320}$$

which can be integrated to get

$$A_m = \frac{2}{(m\pi)^2} + C_2 e^{-(m\pi)^2\mathrm{Fo}} \tag{3.321}$$

where C_2 is an integral constant that needs to be determined by an initial condition. For Fo = 0, eq. (3.312) becomes

$$A_{\mathrm{m}}(0) = 2\int_0^1 \theta(X,0)\cos(m\pi X)dX = 2\int_0^1 \cos(m\pi X)dX = 0 \qquad (3.322)$$

Substituting eq. (3.321) into eq. (3.322), one obtains

$$C_2 = -\frac{2}{(m\pi)^2}$$

therefore, we have

$$A_m = \frac{2}{(m\pi)^2} - \frac{2}{(m\pi)^2} e^{-(m\pi)^2 \mathrm{Fo}}$$

Changing *m* back to *n* for notation,

$$A_n = \frac{2}{(n\pi)^2} - \frac{2}{(n\pi)^2} e^{-(n\pi)^2 \mathrm{Fo}} \qquad (3.323)$$

Substituting eqs. (3.317) and (3.323) into eq. (3.308), the solution becomes

$$\theta(X,\mathrm{Fo}) = \mathrm{Fo} + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos(n\pi X)}{n^2} - \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos(n\pi X)}{n^2}e^{-(n\pi)^2\,\mathrm{Fo}} \qquad (3.324)$$

When the time (Fourier number) becomes large, the last term on the right-hand side will become zero and the solution is represented by the first two terms only. To simplify eq. (3.324), let us assume the solution at large Fo can be expressed as

$$\theta(X,\mathrm{Fo}) = \mathrm{Fo} + f(X) \qquad (3.325)$$

which is referred to as asymptotic solution and it must satisfy eqs. (3.293) – (3.295). Substituting eq. (3.325) into eqs. (3.293) – (3.295), we have

$$f''(X) = 1 \qquad (3.326)$$

$$f'(0) = -1,\ \ f'(1) = 0 \tag{3.327}$$

Integrating eq. (3.326) and considering eq. (3.327), we obtain

$$f(X) = \frac{X^2}{2} - X + C \tag{3.328}$$

where C cannot be determined from eq. (3.327) because both boundary conditions are for the first order derivative. To determine C, we can expand *f*(*X*) defined in eq. (3.328) into cosine Fourier series, i.e.

$$f(X) = \frac{X^2}{2} - X + C = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi X) \tag{3.329}$$

After determining a_0 and a_n, and considering the *f*(*X*) is identical to the second term on the right-hand side of eq. (3.324), we have

$$\frac{X^2}{2} - X + \frac{1}{3} = \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos(n\pi X)}{n^2} \tag{3.330}$$

Substituting eq. (3.330) into eq. (3.324), the final solution becomes

$$\theta(X,\mathrm{Fo}) = \mathrm{Fo} + \frac{X^2}{2} - X + \frac{1}{3} - \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos(n\pi X)}{n^2}e^{-(n\pi)^2\mathrm{Fo}} \quad (3.331)$$

Transient Heat Conduction in a Semi-Infinite Body

Consider heat conduction in a semi-infinite body ($x > 0$) with an initial temperature of T_i. The temperature near the surface of the semi-infinite body will increase because of the surface temperature change, while the temperature far from the surface of the semi-infinite body is not affected and remains at the initial temperature T_i. The physical model of the problem is illustrated in Fig. 3.19

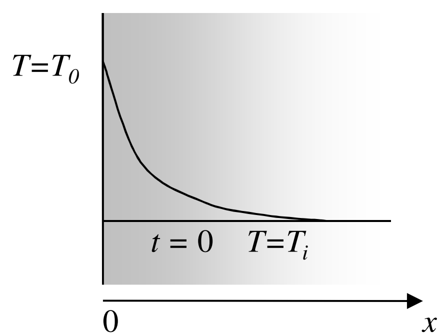

Figure 3.19 Heat conduction in a semi-infinite body

The governing equation of the heat conduction problem and the corresponding initial and boundary conditions are:

$$\frac{\partial^2 T(x,t)}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T(x,t)}{\partial t} \qquad x>0, \quad t>0 \tag{3.332}$$

$$T(x,t) = T_0 \qquad x=0, \quad t>0 \tag{3.333}$$

$$T(x,t) = T_i \qquad x\to\infty, \quad t>0 \tag{3.334}$$

$$T(x,t) = T_i \qquad x>0, \quad t=0 \tag{3.335}$$

which can be solved by using the method of separation of variables or integral approximate solution.

Defining the following dimensionless variables

$$\theta = \frac{T-T_0}{T_i-T_0}, \; X = \frac{x}{L}, \; \mathrm{Fo} = \frac{\alpha t}{L^2} \tag{3.336}$$

where L is a characteristic length.

Eqs. (3.332) – (3.335) will be nondimensionalized as

$$\frac{\partial^2\theta}{\partial X^2} = \frac{\partial\theta}{\partial \text{Fo}}, \quad X>0,\ \text{Fo}>0 \tag{3.337}$$

$$\theta = 0, \quad X = 0 \tag{3.338}$$

$$\theta = 1, \quad \text{X} \to \infty \tag{3.339}$$

$$\theta = 1, \quad X>0,\ \text{Fo}=0 \tag{3.340}$$

Assuming that the temperature can be expressed as

$$\theta(X,\text{Fo}) = \Theta(X)\Gamma(\text{Fo}) \tag{3.341}$$

and substituting eq. (3.341) into eq. (3.337), one obtains

$$\frac{\Theta''(X)}{\Theta(X)} = \frac{\Gamma'(\text{Fo})}{\Gamma(\text{Fo})} = -\lambda^2 \tag{3.342}$$

whose general solutions are:

$$\Theta = C_1 \cos \lambda X + C_2 \sin \lambda X \qquad (3.343)$$

$$\Gamma = C_3 e^{-\lambda^2 \mathrm{Fo}} \qquad (3.344)$$

where C_1, C_2, and C_3 are integral constants.

Substituting eq. (3.341) into eq. (3.338), the following boundary condition of eq. (3.343) is obtained

$$\Theta'(0) = 0 \qquad (3.345)$$

Substituting eq. (3.343) into eq. (3.345) yields and eq. (3.343) becomes

$$\Theta = C_2 \sin \lambda X \qquad (3.346)$$

Substituting eqs. (3.346) and (3.344) into eq. (3.341), the solution becomes

$$\theta_\lambda = C(\lambda) \sin\left(\lambda X\right) e^{-\lambda^2 \mathrm{Fo}} \qquad (3.347)$$

where $C = C_2 C_3$. The general solution for the problem can be obtained by using linear combination of eq. (3.347) for all possible λ, i.e.,

$$\theta = \int_0^\infty C(\lambda)\sin(\lambda X)e^{-\lambda^2 \mathrm{Fo}} d\lambda \qquad (3.348)$$

Substituting eq. (3.348) into eq. (3.340), one obtains

$$1 = \int_{\lambda=0}^\infty C(\lambda)\sin(\lambda X) d\lambda$$

If we solve the problem by using Laplace transformation, we have

$$1 = \int_{\lambda=0}^\infty \sin(\lambda X)\left[\frac{2}{\pi}\int_{X'=0}^\infty \sin(\lambda X')dX'\right]d\lambda$$

Comparing the above two equations, an expression of C is obtained:

$$C(\lambda) = \frac{2}{\pi}\int_{X'=0}^\infty \sin(\lambda X')dX'$$

The temperature distribution, eq. (3.348), becomes

$$\theta = \frac{2}{\pi}\int_{X'=0}^{\infty}\int_{\lambda=0}^{\infty} e^{-\lambda^2 \mathrm{Fo}} \sin(\lambda X') dX' \sin(\lambda X) d\lambda dX'$$

which can be rewritten as

$$\theta = \frac{2}{\pi}\int_{X'=0}^{\infty}\int_{\lambda=0}^{\infty} e^{-\lambda^2 \mathrm{Fo}} \left[\cos\lambda(X - X') - \cos\lambda(X + X')\right] d\lambda dX'$$

Evaluating the integral with respect to yields

$$\int_{\lambda=0}^{\infty} e^{-\lambda^2 \mathrm{Fo}} \cos\lambda(X - X') d\lambda = \sqrt{\frac{\pi}{4\mathrm{Fo}}} \exp\left[-\frac{(X - X')^2}{4\mathrm{Fo}}\right]$$

$$\int_{\lambda=0}^{\infty} e^{-\lambda^2 \mathrm{Fo}} \cos\lambda(X + X') d\lambda = \sqrt{\frac{\pi}{4\mathrm{Fo}}} \exp\left[-\frac{(X + X')^2}{4\mathrm{Fo}}\right]$$

thus

$$\theta = \frac{1}{\sqrt{4\pi \mathrm{Fo}}}\left\{\int_{X'=0}^{\infty} \exp\left[-\frac{(X - X')^2}{4\mathrm{Fo}}\right] dX' - \int_{X'=0}^{\infty} \exp\left[-\frac{(X + X')^2}{4\mathrm{Fo}}\right] dX'\right\} \quad (3.349)$$

Let us define a new variable

$$\eta = \frac{X - X'}{\sqrt{4\mathrm{Fo}}}$$

The first integral in eq. (3.349), becomes

$$\int_{X'=0}^{\infty} \exp\left[-\frac{(X - X')^2}{4\mathrm{Fo}}\right] dX' = \sqrt{4\mathrm{Fo}} \int_{-X/\sqrt{4\mathrm{Fo}}}^{\infty} \exp(-\eta^2) d\eta$$

Similarly, the second integral can be evaluated by following a similar procedure:

$$\int_{X'=0}^{\infty} \exp\left[-\frac{(X + X')^2}{4\mathrm{Fo}}\right] dX' = \sqrt{4\mathrm{Fo}} \int_{X/\sqrt{4\mathrm{Fo}}}^{\infty} \exp(-\eta^2) d\eta$$

Substituting the above two equations into eq. (3.349), we have

$$\theta = \frac{1}{\sqrt{\pi}} \left[\int_{-X/\sqrt{4\mathrm{Fo}}}^{\infty} \exp(-\eta^2) d\eta - \int_{X/\sqrt{4\mathrm{Fo}}}^{\infty} \exp(-\eta^2) d\eta\right] = \frac{2}{\sqrt{\pi}} \int_{0}^{X/\sqrt{4\mathrm{Fo}}} \exp(-\eta^2) d\eta$$

which can be written as

$$\theta = \mathrm{erf}\left(\frac{X}{\sqrt{4\mathrm{Fo}}}\right) \tag{3.350}$$

where erf in eq. (3.350) is the error function defined as:

$$\mathrm{erf}\,(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-z^2}\,dz \tag{3.351}$$

Equation (3.350) can also be rewritten as dimensional form:

$$\frac{T - T_0}{T_i - T_0} = \mathrm{erf}\left(\frac{x}{\sqrt{4\alpha t}}\right) \tag{3.352}$$

The surface heat flux can be obtained by applying the Fourier's law

$$q_0''(t) = -k\left.\frac{\partial T}{\partial x}\right|_{x=0} = \frac{k(T_0 - T_i)}{\sqrt{\pi\alpha t}} \tag{3.353}$$

Periodic boundary conditions can be encountered in various applications ranging from heat conduction in a building during day and night to emerging technologies such as pulsed laser processing of materials. Let us reconsider the problem described by eqs. (3.332) – (3.335) but replace eq. (3.333) by

$$T = T_i + f(t) = T_i + A\cos(\omega t - \beta), \quad x = 0, \quad t > 0 \qquad (3.354)$$

where A is the amplitude of oscillation, ω is the angular frequency, and β is the phase delay. Introducing excess temperature $\vartheta = T - T_i$, the governing equation and corresponding boundary and initial conditions become

$$\frac{\partial^2 \vartheta(x,t)}{\partial x^2} = \frac{1}{\alpha}\frac{\partial \vartheta(x,t)}{\partial t} \qquad x > 0, \quad t > 0 \qquad (3.355)$$

$$\vartheta(x,t) = f(t) = A\cos(\omega t - \beta) \qquad x = 0, \quad t > 0 \qquad (3.356)$$

$$\vartheta(x,t) = 0 \qquad x \to \infty, \quad t > 0 \tag{3.357}$$

$$\vartheta(x,t) = 0 \qquad x > 0, \quad t = 0 \tag{3.358}$$

. Instead of solving eqs. (3.355) – (3.358) directly, we will start with a simpler auxiliary problem defined below:

$$\frac{\partial^2 \Phi(x,t)}{\partial x^2} = \frac{1}{\alpha}\frac{\partial \Phi(x,t)}{\partial t} \qquad x > 0, \quad t > 0 \tag{3.359}$$

$$\Phi(x,t) = f(\tau) = A\cos(\omega\tau - \beta) \qquad x = 0, \quad t > 0 \tag{3.360}$$

$$\Phi(x,t) = 0 \qquad x \to \infty, \quad t > 0 \tag{3.361}$$

$$\Phi(x,t) = 0 \qquad x > 0, \quad t = 0 \tag{3.362}$$

where τ in eq. (3.360) is treated as a parameter, rather than time.

The Duhamel's theorem stated that the solution the original problem is related to the solution of auxiliary problem by

$$\vartheta(x,t) = \frac{\partial}{\partial t}\int_{\tau=0}^{t} \Phi(x,t-\tau,\tau)d\tau \qquad (3.363)$$

which can rewritten using Leibniz's rule

$$\vartheta(x,t) = \int_{\tau=0}^{t} \frac{\partial}{\partial t}\Phi(x,t-\tau,\tau)d\tau + \Phi(x,t-\tau,\tau)\Big|_{\tau=t} \qquad (3.364)$$

The second term on the right hand side is

$$\Phi(x,t-\tau,\tau)\Big|_{\tau=t} = \Phi(x,0,\tau) = 0 \qquad (3.365)$$

therefore, eq. (3.364) becomes

$$\vartheta(x,t) = \int_{\tau=0}^{t} \frac{\partial}{\partial t}\Phi(x,t-\tau,\tau)d\tau \qquad (3.366)$$

The solution of the auxiliary problem can be expressed as

$$\Phi(x,t,\tau) = f(\tau)\left[1 - \operatorname{erf}\left(\frac{x}{\sqrt{4\alpha t}}\right)\right] = \frac{2f(\tau)}{\sqrt{\pi}}\int_{x/\sqrt{4\alpha t}}^{\infty} e^{-\eta^2}\,d\eta \qquad (3.367)$$

The partial derivative appearing in eq. (3.366) can be evaluated as

$$\frac{\partial}{\partial t}\Phi(x,t-\tau,\tau) = f(\tau)\frac{x}{\sqrt{4\pi\alpha}(t-\tau)^{3/2}}\exp\left[-\frac{x^2}{4\alpha(t-\tau)}\right] \qquad (3.368)$$

Substituting eq. (3.368) into eq. (3.366), the solution of the original problem becomes

$$\vartheta(x,t) = \frac{x}{\sqrt{4\pi\alpha}}\int_{\tau=0}^{t}\frac{f(\tau)}{(t-\tau)^{3/2}}\exp\left[-\frac{x^2}{4\alpha(t-\tau)}\right]d\tau \qquad (3.369)$$

Introducing a new independent variable

$$\xi = \frac{x}{\sqrt{4\alpha(t-\tau)}}$$

eq. (3.369) becomes

$$\vartheta(x,t) = \frac{2}{\sqrt{\pi}} \int_{x/\sqrt{4\alpha t}}^{\infty} f\left(t - \frac{x^2}{4\alpha\xi^2}\right) \exp(-\xi^2) d\xi \qquad (3.370)$$

For the periodic boundary condition specified in eq. (3.356), we have

$$\vartheta(x,t) = \frac{2A}{\sqrt{\pi}} \int_{x/\sqrt{4\alpha t}}^{\infty} \cos\left[\omega\left(t - \frac{x^2}{4\alpha\xi^2}\right) - \beta\right] \exp(-\xi^2) d\xi \qquad (3.371)$$

which can be rewritten as

$$\vartheta(x,t) = \frac{2A}{\sqrt{\pi}} \int_{0}^{\infty} \cos\left[\omega\left(t - \frac{x^2}{4\alpha\xi^2}\right) - \beta\right] \exp(-\xi^2) d\xi - \frac{2A}{\sqrt{\pi}} \int_{0}^{x/\sqrt{4\alpha t}} \cos\left[\omega\left(t - \frac{x^2}{4\alpha\xi^2}\right) - \beta\right] \exp(-\xi^2) d\xi \qquad (3.372)$$

Evaluating the first integral on the right hand side of eq. (3.372) yields

$$\vartheta(x,t) = A\exp\left[-x\left(\frac{\omega}{2\alpha}\right)^{1/2}\right]\cos\left[\omega t - x\left(\frac{\omega}{2\alpha}\right)^{1/2} - \beta\right] - \frac{2A}{\sqrt{\pi}}\int_0^{x/\sqrt{4\alpha t}}\cos\left[\omega\left(t - \frac{x^2}{4\alpha\xi^2}\right) - \beta\right]\exp(-\xi^2)d\xi \quad (3.373)$$

It can be seen that as $t \to \infty$, the second term will become zero and the first term represents the steady oscillation.

$$\vartheta_s(x,t) = A\exp\left[-x\left(\frac{\omega}{2\alpha}\right)^{1/2}\right]\cos\left[\omega t - x\left(\frac{\omega}{2\alpha}\right)^{1/2} - \beta\right] \quad (3.374)$$

where $A\exp\left[-x\left(\omega/(2\alpha)\right)^{1/2}\right]$ represents the amplitude of oscillation at point *x*, and $-x\left(\omega/(2\alpha)\right)^{1/2}$ in the cosine function represents the phase delay of oscillation at point *x* relative to the oscillation of the surface temperature.

It is useful here to introduce a concept similar to the thermal boundary layer for convective heat transfer – *thermal penetration depth*. Assuming the thickness of the thermal penetration depth at time t is δ, the temperature of the semi-infinite body at $x > \delta$ will be affected but the temperature at will remain unchanged (see Fig. 3.20).

According to the definition of the thermal penetration depth, the temperature at the thermal penetration depth should satisfy

$$\frac{\partial T(x,t)}{\partial x} = 0 \qquad x = \delta(t) \tag{3.375}$$

$$T(x,t) = T_i \qquad x = \delta(t) \tag{3.376}$$

Integrating eq. (3.332) in the interval (0, δ) one obtains

$$\left.\frac{\partial T}{\partial x}\right|_{x=\delta(t)} - \left.\frac{\partial T}{\partial x}\right|_{x=0} = \frac{1}{\alpha}\int_0^{\delta(t)} \frac{\partial T(x,t)}{\partial t}\,dx \tag{3.377}$$

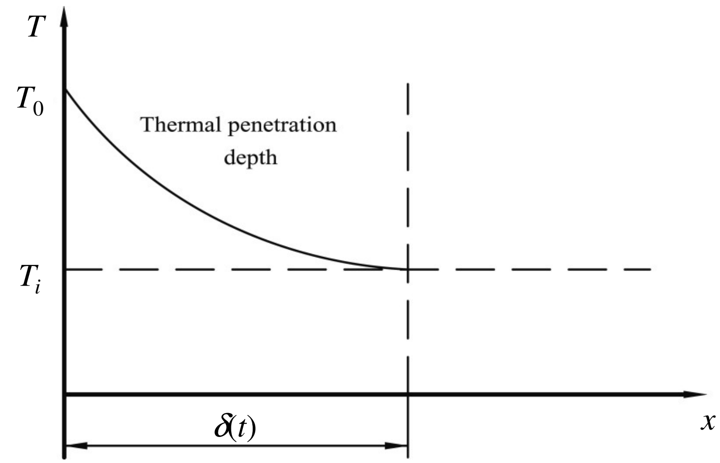

Figure 3.20 Heat conduction in a semi-infinite body with constant wall temperature.

The right-hand side of eq. (3.377) can be rewritten using Leibnitz's rule, i.e.,

$$\left.\frac{\partial T}{\partial x}\right|_{x=\delta(t)} - \left.\frac{\partial T}{\partial x}\right|_{x=0} = \frac{1}{\alpha}\left[\frac{d}{dt}\left(\int_0^\delta T dx\right) - T\big|_{x=\delta}\frac{d\delta}{dt}\right] \qquad (3.378)$$

which represents the energy balance within the thermal penetration depth.

Substituting eqs. (3.375) and (3.376) into eq. (3.378) yields

$$-\alpha \frac{\partial T}{\partial x}\bigg|_{x=0} = \frac{d}{dt}(\Theta - T_i \delta) \tag{3.379}$$

where

$$\Theta(t) = \int_0^{\delta(t)} T(x,t)dx \tag{3.380}$$

Assume that the temperature distribution in the thermal penetration depth is a third-order polynomial function of *x*, i.e.,

$$T(x,t) = A_0 + A_1 x + A_2 x^2 + A_3 x^3 \tag{3.381}$$

where A_0, A_1, A_2, and A_3 and are four constants to be determined using the boundary conditions.

The surface temperature of the semi-infinite body, T_0, is not a function of time *t*, so

$$\frac{\partial T(x,t)}{\partial t} = 0 \qquad x = 0 \tag{3.382}$$

Substituting eq. (3.332) into eq. (3.382) yields

$$\frac{\partial^2 T(x,t)}{\partial x^2} = 0 \qquad x = 0 \tag{3.383}$$

Substituting eq. (3.381) into eqs. (3.333), (3.375), (3.376) and (3.383) yields four equations for the constants in eq. (3.381) . Solving for the four constants and substituting the results into eq. (3.381), the temperature distribution in the thermal penetration depth becomes

$$\frac{T(x,t) - T_i}{T_0 - T_i} = 1 - \frac{3}{2}\left(\frac{x}{\delta}\right) + \frac{1}{2}\left(\frac{x}{\delta}\right)^3 \tag{3.384}$$

where the thermal penetration depth, δ, is still unknown.

Substituting eq. (3.384) into eq. (3.379), an ordinary differential equation for δ is obtained:

$$4\alpha = \delta \frac{d\delta}{dt} \qquad t > 0 \tag{3.385}$$

Since the thermal penetration depth equals zero at the beginning of the heat conduction, eq. (3.385) is subject to the following initial condition:

$$\delta = 0 \qquad t = 0 \tag{3.386}$$

The solution of eqs. (3.385) and (3.386) is

$$\delta = \sqrt{8\alpha t} \tag{3.387}$$

which is consistent with the result of scale analysis, $\delta \sim \sqrt{\alpha t}$

3.3.2 Multidimensional Transient Heat Conduction Systems

Consider transient heat conduction in a rectangular bar with dimensions of $2L_1 \times 2L_2$ and an initial temperature of T_i (see Fig. 3.21). At time $t = 0$, the rectangular bar is immersed into a fluid with temperature T_∞.

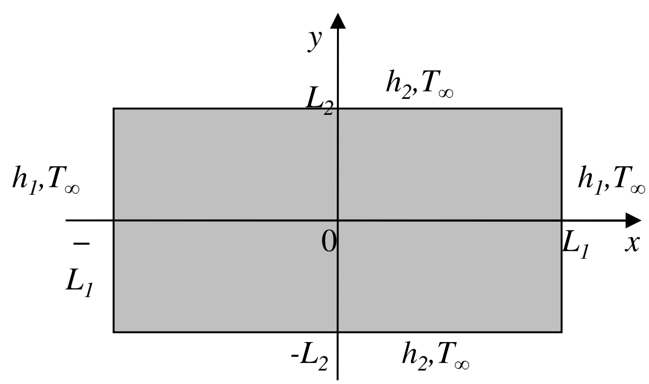

Figure 3.21 Two-dimensional transient heat conduction

The energy equation for this problem is

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=\frac{\partial T}{\partial t},\quad 0<x<L_1,\ 0<y<L_2,\ t>0 \tag{3.388}$$

with the following boundary and initial conditions:

$$\frac{\partial T}{\partial x}=0,\quad x=0,\ 0<y<L_2,\ t>0 \tag{3.389}$$

$$-k\frac{\partial T}{\partial x}=h(T-T_\infty),\quad x=L_1,\ 0<y<L_2,\ t>0 \tag{3.390}$$

$$\frac{\partial T}{\partial y}=0,\quad \mathrm{y}=0,\ 0<x<L_1,\ t>0 \tag{3.391}$$

$$-k\frac{\partial T}{\partial y}=h(T-T_\infty),\quad \mathrm{y}=L_2,\ 0<x<L_1,\ t>0 \tag{3.392}$$

$$T=T_i,\quad 0<x<L_1,\ 0<y<L_2,\ t>0 \tag{3.393}$$

Defining the following dimensionless variables

$$\theta = \frac{T - T_\infty}{T_i - T_\infty}, \ X = \frac{x}{L_1}, \ Y = \frac{y}{L_2}, \ \mathrm{Fo}_1 = \frac{\alpha t}{L_1^2}, \ \mathrm{Fo}_2 = \frac{\alpha t}{L_2^2} \tag{3.394}$$

where both Fo_1 and Fo_2 are Fourier numbers but based on different characteristic lengths. The dimensional time will be related to both Fourier numbers, i.e.,

$$t = t(\mathrm{Fo}_1, \mathrm{Fo}_2) \tag{3.395}$$

therefore, the right-hand side of eq. (3.388) becomes

$$\begin{aligned} \frac{\partial T}{\partial t} &= (T_i - T_\infty)\frac{\partial \theta}{\partial t} = (T_i - T_\infty)\left[\frac{\partial \theta}{\partial \mathrm{Fo}_1}\frac{\partial \mathrm{Fo}_1}{\partial t} + \frac{\partial \theta}{\partial \mathrm{Fo}_2}\frac{\partial \mathrm{Fo}_2}{\partial t}\right] \\ &= (T_i - T_\infty)\left[\frac{\partial \theta}{\partial \mathrm{Fo}_1}\frac{\alpha}{L_1^2} + \frac{\partial \theta}{\partial \mathrm{Fo}_2}\frac{\alpha}{L_2^2}\right] \end{aligned} \tag{3.396}$$

Nondimensionalizing the left-hand side of eq. (3.388) and considering eq. (3.396), the dimensionless energy equation of the problem becomes

$$\frac{\partial^2\theta}{\partial X^2}+\left(\frac{L_1}{L_2}\right)^2\frac{\partial^2\theta}{\partial Y^2}=\frac{\partial\theta}{\partial \mathrm{Fo}_1}+\left(\frac{L_1}{L_2}\right)^2\frac{\partial\theta}{\partial \mathrm{Fo}_2} \quad (3.397)$$

The boundary and initial conditions, eqs. (3.389) – (3.393) are nondimensionalized as

$$\frac{\partial\theta}{\partial X}=0,\quad \mathrm{X}=0,\ 0<Y<1,\ \mathrm{Fo}_1>0,\ \mathrm{Fo}_2>0 \quad (3.398)$$

$$-\frac{\partial\theta}{\partial X}=\mathrm{Bi}_1\theta,\quad \mathrm{X}=1,\ 0<Y<1,\ \mathrm{Fo}_1>0,\ \mathrm{Fo}_2>0 \quad (3.399)$$

$$\frac{\partial\theta}{\partial Y}=0,\quad \mathrm{Y}=0,\ 0<X<1,\ \mathrm{Fo}_1>0,\ \mathrm{Fo}_2>0 \quad (3.400)$$

$$-\frac{\partial\theta}{\partial Y}=\mathrm{Bi}_2\theta,\quad Y=1,\ 0<X<1,\ \mathrm{Fo}_1>0,\ \mathrm{Fo}_2>0 \quad (3.401)$$

$$\theta = 1,\ \ 0 < X < 1,\ \ 0 < Y < 1,\ \ \mathrm{Fo}_1 = \mathrm{Fo}_2 = 0 \tag{3.402}$$

where

$$\mathrm{Bi}_1 = \frac{h_1 L_1}{k},\ \ \mathrm{Bi}_2 = \frac{h_2 L_2}{k} \tag{3.403}$$

are Bio numbers for different surfaces.

The idea of product solution is that the solution of the two-dimensional problem can be expressed as the product of two one-dimensional problem, i.e.,

$$\theta = \varphi(X, \mathrm{Fo}_1)\psi(Y, \mathrm{Fo}_2) \tag{3.404}$$

where φ is the solution of the following problem

$$\frac{\partial^2 \varphi}{\partial X^2} = \frac{\partial \varphi}{\partial \mathrm{Fo}_1} \tag{3.405}$$

$$\frac{\partial \varphi}{\partial X} = 0,\ \ \mathrm{X} = 0,\ \ \mathrm{Fo}_1 > 0 \tag{3.406}$$

$$-\frac{\partial \varphi}{\partial X} = \mathrm{Bi}_1 \varphi, \quad \mathrm{X}=1, \ \mathrm{Fo}_1 > 0 \tag{3.407}$$

$$\varphi = 1, \ 0 < X < 1, \ \mathrm{Fo}_1 = 0 \tag{3.408}$$

and ψ satisfies

$$\frac{\partial^2 \psi}{\partial Y^2} = \frac{\partial \psi}{\partial \mathrm{Fo}_2} \tag{3.409}$$

$$\frac{\partial \psi}{\partial Y} = 0, \quad Y = 0, \ \mathrm{Fo}_2 > 0 \tag{3.410}$$

$$-\frac{\partial \psi}{\partial Y} = \mathrm{Bi}_2 \psi, \quad Y = 1, \ \mathrm{Fo}_2 > 0 \tag{3.411}$$

$$\psi = 1, \ 0 < Y < 1, \ \mathrm{Fo}_2 = 0 \tag{3.412}$$

It can be demonstrated that eqs. (3.397) – (3.402) can be satisfied by eq. (3.404) if φ and ψ are solutions of eqs. (3.405) – (3.408) and (3.409) – (3.4120), respectively. The solution of eqs. (3.405) – (3.408) is:

$$\varphi = \sum_{n=1}^{\infty} \frac{4 \sin \lambda_n}{2\lambda_n + \sin 2\lambda_n} \cos(\lambda_n X) e^{-\lambda_n^2 \mathrm{Fo}_1} \tag{3.413}$$

Where

$$\frac{\lambda_n}{\mathrm{Bi}_1} = \cot \lambda_n \tag{3.414}$$

and the solution of eqs. (3.409) – (3.412) is

$$\psi = \sum_{m=1}^{\infty} \frac{4 \sin \nu_n}{2\nu_m + \sin 2\nu_m} \cos(\nu_m Y) e^{-\nu_m^2 \mathrm{Fo}_2} \tag{3.415}$$

where

$$\frac{\nu_m}{\mathrm{Bi}_2} = \cot \nu_m \tag{3.416}$$

Substituting eqs. (3.413) and (3.415) into eq. (3.404), the solution of the two-dimensional problem is obtained.

$$\theta = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{16 \sin \lambda_n \sin \nu_m \cos(\lambda_n X) \cos(\nu_m Y)}{(2\lambda_n + \sin 2\lambda_n)(2\nu_m + \sin 2\nu_m)} e^{-(\lambda_n^2 \mathrm{Fo}_1 + \nu_m^2 \mathrm{Fo}_2)} \tag{3.417}$$